an algorithmic approach to

MATHEMATICS

cconcrete behavioral foundations

prepared by

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PREFACE

This workbook is designed to parallel the senior author's text, MATHEMATICS: Concrete Behavioral Foundations.

Our purpose in preparing a rather extensive workbook was to make the main ideas in this book more readily available to the beginning student. Many of these ideas are unique, and we feel that they are crucial to effective mathematics instruction in the elementary school. Unfortunately, much of this material might never be acquired by many teachers if they had to rely on the text alone. (The text was written at an intermediate level of difficulty and might, therefore, be rather advanced for the student with no college mathematics.) It has been our experience, nonetheless, that by carefully coordinating the use of the text with the workbook, the major ideas in the text can be made available to even the weakest student. The workbook may be used either to introduce key topics before the student reads the text or to provide highly directed practice after reading. (The text exercises are more varied and diverse and frequently they require more ingenuity on the part of the learner.)

The instructor may want to suggest both of these alternatives to his students and let them select the way which seems to best fit their own learning style. He may also want to experiment with various other ways of coordinating the text and workbook to further achieve his particular goals. We would, in fact, be most interested in hearing from instructors concerning the results of various kinds of experimentation they have tried.

Most of the key ideas in the text have been identified and formulated in the workbook as specific tasks to be performed. Each task is followed by a succinct statement of a rule for solving the task together with several detailed examples showing how the rule can be applied. Hence, the reader may learn to solve the tasks either by reading (and interpreting) explicit rule statements or by inducing (discovering) the underlying rule from examples of its application. The workbook also includes practice exercises for each task.

The approach used in constructing the workbook is based on a theory of mathematical knowledge recently developed by the senior author. This theory goes beyond the usual "behavioral objectives" point of view and has made it possible to build transfer potential directly into the workbook. Thus, the workbook includes certain tasks which may be solved by combining in predetermined ways rules that have been learned earlier. Specifically, we have included certain higher-order rules (marked with an asterisk) which, when applied to other learned rules enable the student to derive new rules for solving new tasks. Tasks which can be solved by using derived rules are checked (✓). The reader should attempt to work these tasks before going to the rule statement. (There are no examples for these tasks.) If unsuccessful, the chances are that one or more of the earlier rules (including higher-order rules) have not been learned.


2The following higher-order rules are included in the workbook: Chap. 4, Rule 10, Rule 11; Chap. 5, Rule 5, Rule 6, Rule 19, Rule 20, Rule 23, Rule 35; Chap. 6, Rule 4, Rule 5; Chap. 8, Rule 63, Rule 64.
In the process of constructing this workbook, an experiment was conducted by Walter Ehrenpreis at Trenton State College as part of his dissertation in which two types of rule-based characterization were contrasted. The discrete characterization consisted of a set of discrete rules which were designed to deal directly with what was in part of the text. The higher-order characterization, which was the main goal of the development, also included (higher-order) rules which reflected relationships among the discrete rules. The introduction of the higher-order rules allowed a sharp reduction (about 50 percent) in the total number of rules, because many of the discrete rules could be derived by use of the higher-order ones. Equally important, a large number of rules that were not dealt with in the original text could also be derived in this way.

The two curricula were compared using a standard experimental procedure and the results were extremely encouraging. Not only did the subjects in the higher-order-rules group have less to learn, but they were able to perform on the untaught rules just as well as the discrete rules group who were taught the rules directly. Furthermore, these subjects were able to solve other tasks (beyond the original text) which the discrete-rules subjects could not.

In order not to confound the results, the evaluation was conducted without the text itself. At first we were apprehensive in doing this. We felt that many of the students might be "turned off" by the task-rule-example-exercise formulation--but things turned out far better than we had thought possible. The reactions of the students (elementary school teachers) were almost uniformly positive even though many of them had never progressed beyond ninth-grade algebra. Most of these students felt that they had learned more mathematics (and in a shorter time) than ever before. (This feeling may have been, in part, because the students knew they were part of an experiment.)

Hawthorne effects notwithstanding, the results suggest that the workbook can be used independently, supplemented only by the instructor's lectures and class discussions. (It might also be used with other textbooks.)

Although those who had most to do with the workbook are listed as co-authors, we owe an important debt to others. Christopher Toy, in particular, helped in formulating Chapters 1 and 2 during the early stages, and we regret that personal reasons made it impossible for him to continue. We would also like to thank our typist, Mrs. Mary Tye, who did such an excellent job on the manuscript. If it were not for her availability, it is unlikely that we would have taken on this task.

We would like to dedicate this workbook to the Mathematics Education Research Group and to those individuals who have helped to further its ideals of research and sound, research-based development in mathematics education.

Joseph M. Scandura

MERG
PART 1
OBJECTIVES AND PROCESS ABILITIES IN MATHEMATICS

CHAPTER 1

Introduction

TASK 1
State two major goals of the recent reform in school mathematics curricula.

ANSWER 1
Three possible answers are:
A. To organize mathematical content about general notions
B. To rid the curriculum of obsolete material
C. To teach children to think creatively

TASK 2
State two current, but extreme approaches to curriculum development.

ANSWER 2
A. Many hold that stating objectives is worthless in educational planning unless they are stated unambiguously and in terms of observable behavior.
B. Others feel that reasoning and other higher level intellectual skills are more important and that they cannot be specified in behavioral terms.
TASK 3

State and define two basic and complementary kinds of abilities associated with knowing mathematics.

ANSWER 3

A. Content abilities: abilities which are directly associated with mathematical content.
B. Processing skills: abilities which are used in doing mathematics, but which are not normally considered to be content.

TASK 4

List six basic kinds of processing skill.

ANSWER 4

A. The ability to detect mathematical regularities, and
B. the reverse ability to particularize.
C. The ability to interpret descriptions of mathematical ideas, and
D. the reverse ability to describe learned mathematical ideas.
E. The ability to make logical inferences, and
F. the reverse ability to axiomatize.
SECTION 1. Discovery -- The Ability to Detect Regularities

TASK 5

Given a set of organized examples, detect the inherent regularities.

The following are examples of detecting regularities in simple situations. The exercises require that you detect regularities in similar situations.

EXAMPLES 5

A. Find a regularity in each sequence:
   1. 1, 2, 4, 8, 16, 32, ...
   2. 1, 4, 9, 16, ...

Answer:
   1. Start with 1, double to get 2. Each term is double the previous term.
   2. The terms are the squares of the consecutive positive integers.

B. Find a regularity
   1 + 3 = 4 = 2^2
   1 + 3 + 5 = 9 = 3^2
   1 + 3 + 5 + 7 = 16 = 4^2

Answer:
   The sum of the first n odd positive integers is n x n or n^2.

C. Find a regularity in the sequence:
   ↑, ↓, ↑, ↓, ↑, ...

Answer:
   Begin with an arrow pointing up. Successive terms alternate -- one arrow down, one arrow up.

D. Find two different regularities in each of the following displays.
   1. ○ ○ □ ○ ○ ...
   2. 1, 2, 4, ...
Answer:

1. The sequence may be seen as repeating "O O G n" as in:

O O G O O G O O G O O G
or as repeating "O O G O" to give
O O G O O G O O G O O G

2. Two possible regularities are:

(a) Start with 1. Each term is the double of the previous one, i.e., 1, 2, 4, 8, 16, ...

(b) Start with 1. Add n to get the (n + 1)th term, i.e., 1, 2, 4, 7, 11, 16, ...

E. Find a regularity common to the displays:

1. \[
\begin{array}{ccc}
6 & 7 & 2 \\
3 & 8 & 4 \\
7 & 5 & 3 \\
\end{array}
\]

2. \[
\begin{array}{ccc}
2 & 5 & 6 & 7 \\
7 & 4 & 8 & 1 \\
5 & 1 & 3 & 11 \\
1 & 14 & 2 & 3 \\
\end{array}
\]

Answer:

1. The sum of the numbers in each row of square 1 is the same, 15.
2. The sum of the numbers in each row of square 2 is the same, 20.

EXERCISES 5

The regularities in each exercise parallel those of the correspondingly numbered example.

A. Find a regularity in each sequence:

1. 1, 3, 9, 27, ...
2. 1, 8, 27, 64, ... (Hint: consider cubes of natural numbers).

B. Find a regularity:

The number of subsets of a set with 1 element is \(2 = 2^1\)
with 2 elements is \(4 = 2^2\)
with 3 elements is \(8 = 2^3\)
C. Find a regularity in the sequence:

\[ \uparrow, \rightarrow, \downarrow, \leftarrow, \uparrow, \rightarrow, \downarrow, \leftarrow, \uparrow, \ldots \]

D. Find two different regularities in each of the displays:

1. \[ \ast \ \Delta \ \ast \ \Delta \ \ast \ \cdots \]

2. \[ 2, 3, 5, 8, \_, \_, \_, \_, \_, \ldots \]

(Hints for 2: \(2 + ? = 3, 3 + ? = 5, 5 + ? = 8\). What two numbers have a sum of 5? of 8?)

E. Find a regularity common to the displays:

\[
\begin{array}{ccc}
6 & 5 & 2 \\
1 & 3 & 4 \\
8 & 7 & 9
\end{array}
\] 

\[
\begin{array}{cccc}
1 & 6 & 14 & 9 \\
12 & 15 & 7 & 4 \\
8 & 3 & 2 & 16 \\
13 & 10 & 11 & 5
\end{array}
\]

---

TASK 6

Detect regularities in given displays by applying particular processing skills (e.g., by organizing displays).

PROCEDURE 6

Apply one of the specific processing skills (illustrated in the examples). Then try to detect regularities.

EXAMPLES 6

A. Find a regularity in the display:

\[
\begin{array}{ccc}
1 + 3 + 5 &= 9 \\
1 &= 1 \\
1 + 3 + 5 + 7 &= 16 \\
1 + 3 &= 4 \\
1 + 3 + 5 + 7 + 9 &= 25 \\
\ldots &= \ldots \\
\ldots &= \ldots
\end{array}
\]
The processing skill of rearranging displays suggests that we order the series according to the number of terms.

\[
\begin{align*}
1 &= 1 \\
1 + 3 &= 4 \\
1 + 3 + 5 &= 9 \\
1 + 3 + 5 + 7 &= 16 \\
1 + 3 + 5 + 7 + 9 &= 25
\end{align*}
\]

B. Consider the preceding display. Find a regularity in the sequence of sums.

Answer:

Examine the sequence of sums 1, 4, 9, 16, 25, ..., By recalling a previously learned regularity for this display (c.f., TASK 5, Example A2), we see that the sequence is the same as \(1^2, 2^2, 3^2, 4^2, 5^2\). Hence,

\[
\begin{align*}
1 &= 1^2 \\
1 + 3 &= 2^2 \\
1 + 3 + 5 &= 3^2 \\
1 + 3 + 5 + 7 &= 4^2 \\
1 + 3 + 5 + 7 + 9 &= 5^2 \\
&\cdot \\
&\cdot \\
&\cdot 
\end{align*}
\]

C. Find a regularity in the following display:

Answer:

Find all the symmetry lines of a square, i.e., lines of reflection which leave the figure unchanged. For the square they are:

1. 2. 3. 4.

See if rotating about one of these lines will give the display of example three. Trying each symmetry line in turn, we discover that 3 is the appropriate line for
EXERCISES 6

A. Use "rearranging" to order the following display, then detect the regularity:

<table>
<thead>
<tr>
<th>3</th>
<th>10</th>
<th>7</th>
<th>4</th>
<th>9</th>
<th>2</th>
<th>5</th>
<th>1</th>
<th>8</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

(Hint: order the left column of numbers.)

B. The number of subsets of a set with

<table>
<thead>
<tr>
<th>Elements</th>
<th>Subsets</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
</tr>
</tbody>
</table>

How many subsets does a set of 5 elements have? of 6? of n elements? (Hint: See Exercise 5B.)

C. Find the regularity in the display:

\[
\begin{align*}
\triangle & \rightarrow \bigtriangleup \\
\bigtriangleup & \rightarrow \triangle \\
\bigtriangleup & \rightarrow \bigtriangleup
\end{align*}
\]
TASK 7

Describe what the ability to detect regularities involves.

ANSWER 7

Detecting a regularity involves perceiving a pattern or abstracting what is common to a number of examples. This process is often called reasoning by induction.

TASK 8

Describe an important processing skill and tell why it is important.

ANSWER 8

Three possible answers are:
A. Organizing or rearranging displays — putting a display in some ordered form may facilitate using an already learned rule, or recognition of a new one.
B. Scanning selectively — scanning a display with some rule in mind may facilitate determining whether the display includes that rule.
C. Looking for lines of symmetry — in geometric displays, lines of symmetry may reveal rules which are based on symmetric properties of a display.

TASK 9

Give two ways of determining if a pupil is able to detect a regularity.

ANSWER 9

A. See if the pupil can give the correct response to previously unseen examples of the same regularity.
B. See if the pupil can describe in words the regularity he has found.

EXAMPLES 9

A. Consider the sequence

6, 2, 5, 1, 4, 0, ...

Answer:

The pupil can demonstrate his knowledge of the regularity by:
1. Stating, "Beginning with 6, alternately subtract 4 and then add 3 to find the consecutive terms."
2. Giving the next few terms, e.g.,

   \[6, 2, 5, 1, 4, 0, 3, -1, 2, -2, \ldots\]

B. Consider the sequence:

   \[\uparrow, \rightarrow, \rightarrow, \uparrow, \rightarrow, \rightarrow, \ldots\]

Answer:

   The pupil can demonstrate his knowledge of the regularity by:
   1. Stating the pattern "arrow pointing up, two arrows pointing right."
   2. Giving the next few terms of the sequence, e.g.,

   \[\uparrow, \rightarrow, \rightarrow, \uparrow, \rightarrow, \rightarrow, \uparrow, \rightarrow, \rightarrow, \rightarrow, \rightarrow, \ldots\]

C. Consider the magic square:

   \[
   \begin{array}{ccc}
   6 & 7 & 2 \\
   1 & 5 & 9 \\
   8 & 3 & 4 \\
   \end{array}
   \]

Answer:

   The pupil can demonstrate his knowledge by:
   1. Stating, "The sum of the numbers in each row, each column, and each
      major diagonal (dotted diagonals) is the same, 15.
   2. Generating a related new square (New magic squares may be constructed by
      adding or multiplying every entry in the square by the same number.)

   \[
   \begin{array}{ccc}
   12 & 14 & 4 \\
   2 & 10 & 18 \\
   16 & 6 & 8 \\
   \end{array}
   \]

EXERCISES 9

   Give two ways a pupil might demonstrate that he has detected a regularity in each
   of the following:
A. \( \circ, \bigcirc, \square, \bigcirc, \circ, \bigcirc, \square, \ldots \)

B. \( 1, 1, 2, 3, 5, 8, 13, \ldots \)

C. 

<table>
<thead>
<tr>
<th>18</th>
<th>21</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>15</td>
<td>27</td>
</tr>
<tr>
<td>24</td>
<td>9</td>
<td>12</td>
</tr>
</tbody>
</table>

TASK 10

Given a regularity (i.e., idea or rule) construct a task requiring a pupil to detect the regularity for himself.

PROCEDURE 10

Construct a sequence of examples (instances) of the given regularity (rule) which could lead a pupil to detect that rule or regularity.

EXAMPLES 10

A. Given the regularity, "The sum is equal to the square of the number of terms," construct a task requiring a pupil to detect this regularity.

Answer:

Detect a regularity inherent in:

\[
1 = 1 \\
1 + 3 = 4 \\
1 + 3 + 5 = 9
\]

B. Given the regularity, "The first term is two. Each successive term is obtained by tripling the preceding term," construct a task requiring a pupil to detect this regularity.

Answer:

Detect the regularity inherent in sequence 1:

1. \( 2, 6, 18, 54, 162, \ldots \)

Notice that sequences 2 and 3 have essentially the same regularity. The only difference is that sequence 2 starts with -1 and sequence 3, with 5.

2. \( -1, -3, -9, -27, -81, \ldots \)

3. \( 5, 15, 45, 135, \ldots \)
C. Given the regularity, "In certain 3x3 squares, called magic squares, the sum of the numbers in each row, each column, and each major diagonal is the same," construct a task requiring a pupil to detect this (common) regularity.

Answer:

Detect the common regularity inherent in the following 3x3 squares:

1. \[
\begin{array}{ccc}
7 & 8 & 3 \\
2 & 6 & 10 \\
9 & 4 & 5 \\
\end{array}
\]

2. \[
\begin{array}{ccc}
4 & 5 & 0 \\
-1 & 3 & 7 \\
6 & 1 & 2 \\
\end{array}
\]

3. \[
\begin{array}{ccc}
14 & 16 & 6 \\
4 & 12 & 20 \\
18 & 8 & 10 \\
\end{array}
\]

D. Given the regularity, "Continue to repeat the first three elements of the sequence in the order they are given," construct a task requiring a pupil to detect this common regularity.

Answer:

Detect the common regularity inherent in the following sequences:

1. \( *, \bigcirc, \Delta, *, \bigcirc, \Delta, *, \bigcirc, \Delta, \ldots \)

2. \( a, b, c, a, b, c, a, b, c, \ldots \)

3. \( 2, 4, 6, 2, 4, 6, 2, 4, 6, \ldots \)

EXERCISES 10

A. Given the regularity, "Start with 1, 2. Then obtain each succeeding term by multiplying the preceding two terms," construct a task requiring a pupil to detect this regularity. (Hint: What term follows 1, 2, ? Fill in the blanks 1, 2, ____, ____, ____, ____, ...)

B. Given the regularity, "Start with any two numbers. Then obtain each succeeding term by multiplying the preceding two terms," construct a task (i.e., some sequences) requiring a pupil to detect this (common) regularity.

C. Given the regularity, "In certain 4x4 squares the sum of numbers in each row, each column, and each major diagonal is the same," construct a task requiring a pupil to detect this regularity. (Hint: The 4x4 square on the following page is one which has the above regularity).
D. Construct a task requiring a pupil to detect a regularity common to one in the display:

1, 0, 1, 0, 0, 1, 0, 0, 0, 1, ...

E. Construct a task requiring a pupil to detect a regularity common to one in the display:

* , O , † , X , * , O , † , X , ...
SECTION 2. Particularization -- The Ability to Construct Examples

TASK 11

Particularize specific kinds of simple ideas or regularities.

The following are examples of particularizing regularities in simple situations. The exercises require you to particularize regularities in similar situations.

EXAMPLES 11

A. Construct a pair of natural numbers with the second number double the first, i.e., a pair of the form (x, 2x).

Possible Answers:

(5, 10), (6, 12), (10, 20), etc.

B. Give examples of the sequence x, y, z, x, y, z, ... where x, y, z can be any objects.

Possible Answers:

The rule requires us to repeat the pattern "x, y, z" whatever x, y, z may be. Choose objects for x, y, z.

1. Let x be "red," y be "green," z be "yellow." Then the sequence is the traffic light sequence.

red, green, yellow, red, green, yellow, ...

2. Let x = 1, y = 2, z = 3. Then the sequence is

1, 2, 3, 1, 2, 3, ...

3. Let x = □, y = △, z = ○. Then the sequence is

□, △, ○, □, △, ○, ...

C. Modify an example of a rule to give new examples. Consider the array

1
5 6 2
10

The product of all elements of the row is the same as the product of all elements of the column, namely, 60.
Answer:

By multiplying every element by the same number, we obtain a new array following the same rule. For example, multiplying by 2, we get:

\[
\begin{array}{c}
2 \\
10 & 12 & 4 \\
20
\end{array}
\]

Check that the products are equal.

EXERCISES 11

A. Construct pairs of numbers with the second number the cube of the first, i.e.,

\[(x, x^3)\]

B. Construct three new sequences having the same regularity as

a, b, c, a, a, b, b, c, c, a, a, a, ...

C. Note that the numbers at the opposite ends of the lines in the following display add up to a fixed sum.

\[
\begin{array}{c}
9 \\
7 \\
5 \\
3 \\
1 \\
2 \\
4 \\
6 \\
10 \\
8
\end{array}
\]

Give a second example of this regularity.

TASK 12

Describe the ability to particularize and state its relation to detecting regularities.
The ability to particularize is the inverse of the process of detecting regularities. In detecting regularities, one determines a regularity from given examples. In particularization one generates new examples of given regularities.

**TASK 13**

Describe two ways of finding out if a pupil can particularize.

**ANSWER 13**

(1) See if the pupil can describe verbally the method used to generate examples, e.g., substitution, analogy, etc.

(2) See if the pupil can construct examples of given (statements of) regularities.

**EXAMPLES 13**

A. If a pupil has detected the regularity underlying the display:

\[
\begin{align*}
2 & \rightarrow 7 \\
3 & \rightarrow 10 \\
4 & \rightarrow 13
\end{align*}
\]

describe two ways in which he might demonstrate his ability to construct new examples of this regularity.

Possible Answers:

(1) The pupil says "Take a number, say 5; triple it, giving 15; add one to the result, giving 16."

(2) The pupil constructs the instance \(5 \rightarrow 16\).

B. If the pupil is given the regularity, "Take three symbols. Write the first, then the second, then the first again. Then write the third. Repeat this pattern." Describe two ways in which he might demonstrate his ability to generate new examples of this regularity.

Possible Answers:

(1) The pupil says, "Let \(x\) be the first symbol, \(y\) the second and \(z\) the third. Then, the pattern is \(x, y, x, z, x, y, x, z,\ldots\)."

(2) The pupil constructs: \(\Delta, \circ, \Delta, \Box, \Delta, \circ, \Delta, \Box, \ldots\)
C. If the pupil is presented with the rule \(a + b + c = b + d + f = e + f + g\) for a figure of the form

\[
\begin{array}{ccc}
a & e \\
b & d & f \\
c & g \\
\end{array}
\quad \text{and the example}
\begin{array}{ccc}
3 & 3 \\
4 & 6 & 2 \\
5 & 7 \\
\end{array}
\]

Describe two ways in which the pupil might demonstrate his ability to particularize.

Possible Answers:

1. He says, "Find numbers such that the sum of the elements in each of the vertical columns and the horizontal crossbar is fixed."
2. The pupil constructs:

\[
\begin{array}{ccc}
2 & 2 \\
3 & 5 & 1 \\
4 & 6 \\
\end{array}
\]

EXERCISES 13

In each of the following, describe two ways in which the pupil might demonstrate his ability to particularize.

A. The pupil is presented with the rule, "To each natural number \(x\), assign the number \(3x - 1\)."

B. The pupil is presented with the rule, "Take three symbols. In turn, repeat each one twice. Continue with the same pattern."

C. The pupil is presented with the rule \(a + b + c = a + d + e = c + h + g = e + f + g\), for a figure of the form

\[
\begin{array}{ccc}
a & b & c \\
d & h \\
e & f & g \\
\end{array}
\quad \text{and the example}
\begin{array}{ccc}
3 & 10 & 6 \\
8 & 3 \\
8 & 1 & 10 \\
\end{array}
\]
SECTION 3. The Ability to Interpret Mathematical Descriptions

TASK 14

Interpret verbal, iconic, and symbolic descriptions of simple ideas or regularities.

Each of the following examples describes an idea or regularity in verbal, iconic, and symbolic form. The exercises involve interpreting similar descriptions.

EXAMPLES 14

A. 1. Verbal Description: To add four whole numbers, start from the left and add the first two numbers. Then, take that sum and add the third number. Take this new sum and add the fourth number. The last sum obtained is the sum of the four numbers.

Example:

Suppose the numbers are 3, 2, 1, 6. First, add 3 and 2. The answer is 5. Then add 5 to the third number, 1. The answer is 6. Finally, add 6 to the fourth number, 6, giving 12.

2. Iconic Description: The sum of four numbers, a, b, c, and d is

```
  a  b  c  d
```

Count the length of the line segment obtained to get a + b + c + d

Example:

Suppose the numbers are 1, 3, 4, 2. First, translate the numbers into line segments giving:
Then put the segments end to end.

The sum is ten.

3. Symbolic Description: To add four whole numbers, $a$, $b$, $c$, $d$, compute

$$(((a + b) + c) + d)$$

Example:

Suppose $a$, $b$, $c$, $d$ are 1, 2, 3, 4 respectively. Following the arbitrary convention that operations within the innermost parentheses are performed first, we get

$$(((1 + 2) + 3) + 4)$$

$$= (( 3 + 3) + 4)$$

$$= ( 6 + 4)$$

$$= 10$$

B. 1. Verbal Description: To help Bob find John's house, John tells him:

Go out your front door; walk to the left; at the second cross street turn right; my house is the second one on the left.

(To successfully interpret the description, Bob must know the meanings of the words "left," "right," "two," "cross street, etc." He must also be able to integrate these meanings into a coherent whole.)

2. Iconic Description: John gives Bob the map:
(To interpret the map, Bob must recognize that the squares represent blocks separated by streets, the small squares represent houses on a block, and the path he is to follow is marked by the arrow.)

3. Symbolic Description: John gives Bob a piece of paper with the symbols (L, 2, R, 2, L)

(To interpret this symbol, Bob must know that "L" means "Turn left," "R" means "Turn right," "2" tells how many cross streets or houses to "go." He must also know that the symbols must be read from left to right.)

C. 1. Verbal Description: Suppose the annual budget of a school system is divided into four equal parts. Then, two parts would be for salaries, and the remaining two parts for buildings and miscellaneous expenses, respectively.

(For example, if the total budget for a school system were one million dollars, each part would represent a quarter of a million dollars. Hence, the amount budgeted for salaries would be five hundred thousand dollars.)

2. Iconic Description:

![School District Budget Diagram]

3. Symbolic Description:

<table>
<thead>
<tr>
<th>Description</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Salaries</td>
<td>50% of total</td>
</tr>
<tr>
<td>Buildings</td>
<td>25% of total</td>
</tr>
<tr>
<td>Miscellaneous</td>
<td>25% of total</td>
</tr>
</tbody>
</table>

**EXERCISES 14**

A. In Example 1, find the sum of the numbers 5, 3, 2 and 2 using:

1. the verbal description;
2. the iconic description;
3. the symbolic description.

B. 1. Given the following directions from Jeanne's house to Julie's house, draw the correct path on the block diagram beneath the directions.
"Leave Jeanne's house. Turn right. At the first cross street turn left and cross two more streets. Julie's house is the third house on the right."

2. Given that the two following symbolic descriptions start at the same place, see if they end at the same place.

(L, 3, R, 2, R, 2, R)

(R, 2, L, 3, L, 3, L, 1, L, 2, R)

(Hint: Draw an iconic map as in Example B2 above.)

C. 1. The cost of regular gasoline is divided into three equal parts, with one of the parts being for tax. If a person buys six dollars worth of regular gasoline, how much is the tax.

2. The picture below is an iconic description of how much of the cost of regular gasoline is for tax. Is it an accurate icon?

3. Calculate how much of $70 spent for regular gasoline is for tax.

____________

TASK 15

Describe generally the ability to interpret descriptions of mathematical ideas or regularities.

ANSWER 15

The ability to interpret involves determining the meaning of descriptions, which may (a) be given orally or in written form, (b) involve mathematical symbols or words, (c) be of a pictorial (iconic) nature or otherwise. By the meaning of a description
is meant the idea, rule, or rules the description denotes.

_______ · _______

**TASK 16**

Give two ways of determining if a pupil can interpret a description.

**ANSWER 16**

1. See if the pupil can apply the rule to a particular instance.
2. See if the pupil can describe the meaning of the description in his own words (i.e., if he can give a *paraphrase*).

**EXAMPLES 16**

A. Description: The area of a triangle is one half the base times the height.

Answer:

1. Find the area of a triangle that has a height of 3" and a base of 4".  
   (The answer is $1/2 \times 4 \times 3 = 6$.)

2. State the rule for finding the area of the triangle in your own words.  
   (One possible answer is: "Multiply the base and height and then divide by two."

B. Description: For all natural numbers $m$, $n$, $m + n = n + m$.

Answer:

1. Apply the rule for $m = 2$, $n = 3$.  
   (The answer is $2 + 3 = 3 + 2 = 5$.)

2. State the rule in your own words.  
   (One possible answer is: "You can add in either order and get the same answer.")

C. Description: $(m + n)^2 = m^2 + 2mn + n^2$ where $m$ and $n$ are real numbers.

Answer:

1. Find $(2 + 3)^2$.  
   (The answer is: $(2 + 3)^2 = 2^2 + 2 \cdot 2 \cdot 3 + 3^2 = 4 + 12 + 9 = 25$.)

2. What does $(m + n)^2 = m^2 + 2mn + n^2$ mean?  
   (One possible answer is: "The square of the sums of $m$ and $n$ is the sum of the square of $m$, the square of $n$, and the product of 2, $m$, and $n$."

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EXERCISES 16

A. Description: The area of a rectangle is the base times the height.

B. Description: For all natural numbers $l$, $m$, $n$, $(l \cdot m) \cdot n = l \cdot (m \cdot n)$

C. Description: For all natural numbers $m$, $(m + 1)(m - 1) = m^2 - 1$. 

_______ $\cdot$ ______
SECTION 4. The Ability to Describe Mathematical Ideas

TASK 17

Given a mathematical idea, or regularity, describe it in verbal, iconic, or symbolic form.

EXAMPLES 17

A. Give verbal and symbolic descriptions of:

\[ \uparrow, \downarrow, \downarrow, \uparrow, \downarrow, \downarrow, \uparrow, \ldots \]

Verbal Description:

Draw an arrow pointing up, then to the right draw two arrows pointing down. Continue the pattern: one arrow up, then two arrows down.

Symbolic Description:

\[ x, y, y, x, y, y, \ldots; \ x = \uparrow, \ y = \downarrow. \]

B. Give symbolic and iconic descriptions of:

1. It is quarter to one.
2. It is twelve forty-five.
3. It is five-twenty-five.

Iconic Description:

1. \( \circled{\_!} \)
2. \( \circled{\_!} \)
3. \( \circled{-} \)

Symbolic Description

1. 12:45
2. 12:45
3. 5:25
C. Find the area of a trapezoid.

Let b and b' be the bases and h be the height of a trapezoid. Give verbal and iconic descriptions of:

\[
\text{Area} = \frac{1}{2} bh + \frac{1}{2} bh'
\]

Verbal Description:

The area of a trapezoid is one-half of the product of the height (i.e., the distance between the two bases) and the length of one of the bases plus one-half of the product of the height and the other base.

Iconic Description:

\[
\begin{array}{c}
\text{b} \\
\hline
\text{h} \\
\text{b'} \\
\end{array}
\quad =
\begin{array}{c}
\text{b} \\
\hline
\text{b'} \\
\end{array}
\quad +
\begin{array}{c}
\text{h} \\
\hline
\end{array}
\]

EXERCISES 17

A. Give verbal and symbolic descriptions of:

\[
\uparrow, \downarrow, \longrightarrow, \uparrow, \downarrow, \longrightarrow, \ldots
\]

B. Give symbolic and iconic descriptions of: North, West, South, East, North,…

C. Let b be the base and h be the height of a parallelogram. Give verbal and iconic descriptions of Area = bh.

_______ · _______

TASK 18

Discuss briefly the ability to describe mathematical regularities.

ANSWER 18

Describing a mathematical idea or regularity involves representing the idea or regularity in verbal, symbolic, or iconic form, hopefully in a way which is meaningful to some other person. (By interpreting such a description, the other person may acquire knowledge of the original regularity. Note that knowledge of a regularity does not necessarily imply the ability to describe the regularity.)

_______ · _______

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TASK 19

Judge and correct descriptions of a given idea or regularity.

ANSWER AND PROCEDURE 19

Judge and correct descriptions according to the following four basic criteria.

A. Descriptions must be composed of words, symbols, or icons whose meanings are known to the hearer or reader and which are combined in a grammatical way.

B. Descriptions must completely characterize the idea to be described.

C. Descriptions must accurately characterize the idea to be described.

D. Descriptions must be concise.

EXAMPLES 19

A. The area of a triangle is $1/2 \, bh$.

Correction: $b$ and $h$ are symbols whose meanings are not necessarily known. Improve the description by adding, "where $b$ is the base and $h$ is the height of the triangle."

B. Even numbers are integers.

Correction: The statement is incomplete. There are integers which are not even numbers (e.g., 1, 3, 7). The simplest correction is to add "which are divisible by two" to the end of the statement.

C. Suppose a natural number $n$ divides $a \times b$, where $a$ and $b$ are natural numbers. Then $n$ divides $a$ or $b$.

Correction: This description is inaccurate since 4 divides $2 \times 6 = 12$ and 4 does not divide either 2 or 6. The quickest correction is to require that $n$ be a prime number.

D. Consider the statement, "A prime number is any natural number other than one whose only divisors are one and itself." This is complete and accurate.

There is a more concise definition: "A prime number is a natural number with exactly two (distinct) divisors." (Check that this definition includes the stipulation that the number one is not a prime.)

EXERCISES 19

A. The area of a circle is $\pi r^2$.

B. Quadrilaterals are four sided figures with opposite sides equal.

C. A square is a quadrilateral with four equal sides.

D. Two distinct and different points determine a straight line.
SECTION 5.  The Ability to Make Logical Inferences

TASK 20

Given a set of statements (premises), draw a simple (logical) inference.

PROCEDURE 20

Determine what statement (conclusion) follows naturally from them.

EXAMPLES 20

A. If the teacher is absent tomorrow, then we will have a substitute. Tomorrow the teacher will be absent.

Conclusion:

Tomorrow we will have a substitute.

B. If it is raining, then the ground is wet. It is raining.

Conclusion:

The ground is wet.

Comment:

Examples A and B follow the logical rule modus ponens. In this rule if the two premises are:
1. If P then Q,
2. P,

then the conclusion is:

3. Q.

(P and Q represent statements.)

C. Either John is here, or he has already left for the game. John is not here.

Conclusion:

John has already left for the game.

D. You must purchase your ticket either at school or at the game. You are not able to purchase your ticket at school.

Conclusion:

You must purchase your ticket at the game.
Examples C and D follow the logical rule of disjunctive syllogism. In this rule if the two premises are:
1. P or Q.
2. Not P (or not Q),
then the conclusion is:
3. Q (or P).
(P and Q represent statements.)

E. All men must someday die. John is a man.

Conclusion:
John must someday die.

F. All multiples of 4 are even numbers. 16 is a multiple of 4.

Conclusion:
16 is an even number.

Comment:
Examples E and F follow the logical form (often called syllogistic reasoning):
1. All P's are Q's.
2. x is a P.

The conclusion is:
C. x is a Q.

(P and Q stand for an abstract noun or property. x is a particular noun or property.)

EXERCISES 20

Determine a natural conclusion for each set of premises. State the type of logical deduction used.

A. All Dodge cars are built by the Chrysler Motor Company. This car is a Dodge.
B. Either Mary is home, or she went to stay with her Aunt. Mary is not home.
C. Each whole number corresponds to an integer. 6 is a whole number.
D. If the bell has rung, then we are late. The bell has rung.
E. Either the clock is fast, or we are early for our appointment. We are on time (not early) for our appointment.
F. If the janitor is here, then the mess will be cleaned. The janitor is here.
TASK 21

Describe the ability to make logical inferences.

ANSWER 21

The ability to make a logical inference involves deducing new ideas (rules) from a set of given ideas (rules). This ability is sometimes called deductive reasoning.

TASK 22

Illustrate what a person must do to show that he has the ability to make logical inferences.

ANSWER 22

When presented with a set of premises he must be able to draw the proper conclusion (if any). If given an additional statement he must be able to tell whether or not the given statement follows logically from the premises, is independent of them, or contradicts them.

EXAMPLES 22

The following examples include both the tasks and possible correct answers.

A. A person is given the premises: All drivers in this state have operating licenses. John drives in this state.

He concludes:

John has an operating license.

B. A person is given the premises: All athletes in this tournament are amateurs. John is an amateur.

And the conclusion:

John is in this tournament.

He answers: The conclusion "John is in this tournament," is an invalid use of syllogistic reasoning. John may be an amateur but not be in this tournament.

C. A person is given the premises: If it is not raining, then George will go to the game. It is not raining.

And the conclusion:

George will go to the game.
He answers: The reasoning is valid. (The logical form is modus ponens.)

D. A person is given the premises: If x is a square, then x is a rectangle. x is a rectangle.

He says: I cannot draw a (natural) conclusion from these premises.

E. A person is given the premises: Either the lion is not feeling well, or it has not been fed. The lion has been fed.

He concludes: The lion is not feeling well.

F. A person is given the premises: Either Jane was sick, or she did not pass the course. Jane was sick.

And the conclusion:

Jane passed the course.

He answers: The conclusion "Jane passed the course" is an invalid use of the logical form: P or Q, not P (or not Q). Jane might have been sick and also not passed the course. With this inference rule "either P or Q" means "either P, Q, or P and Q."

EXERCISES 22

The person is given the following sets of premises, or premises and conclusions. Give a response which indicates that he has the ability to make a logical inference.

A. All Beagles are dogs. Fido is a Beagle.
B. I will pass the course, or I will quit school. I will not quit school.
Therefore I will pass the course.
C. All snakes are reptiles. A cobra is a snake.
D. If it is raining, then the ground is wet. The ground is wet. Therefore it is raining.
E. If I will pay five dollars, then I will get a ticket. I will pay five dollars.
F. Either I forgot where I parked the car, or it was stolen. The car was stolen.
Therefore I did not forget where I parked the car.

TASK 23

Given an abstract (symbolic) logical form, find a set of verbal statements which serve as premises requiring use of logical inferences for that form.

PROCEDURE 23

Substitute phrases, nouns, properties and/or sentences for the symbols in the premises of the logical form.
EXAMPLES 23

A. Modus ponens has the form:
   If P then Q.
   P.
   Therefore Q.

Substitute for P the statement,
   John is in a hurry, and for Q,
   John will skip dinner.

The logical form modus ponens then gives the premises:
   If John is in a hurry, then John will skip dinner. John is in a hurry.

B. Simple syllogistic reasoning has the form:
   All P's are Q's.
   x is a P.
   Therefore x is a Q.

Substitute the noun, men, for P and the property, mortal, for Q and the specific name, Socrates, for x. This gives the premises:
   All men are mortal. Socrates is a man.

C. Disjunctive syllogism has the form:
   Either P or Q.
   Not P (or, not Q).
   Therefore Q (or, therefore P).

Substitute for P the statement,
   John is a doctor, and for Q,
   John is a lawyer.

This gives the premises:
   Either John is a doctor or John is a lawyer. John is not a lawyer.

EXERCISES 23

Construct another set of premises for each of the three logical forms above.
SECTION 6. The Ability to Axiomatize

TASK 24

Given a set of statements, determine which statements may be viewed as premises and which may be deduced.

PROCEDURE 24

Determine which statements are logically dependent, i.e., can be inferred from the others. These may be viewed as conclusions; the others, as premises.

EXAMPLES 24

A. 1. Joe is Mary’s husband.
   2. Mary is Joe’s wife.
   3. Tommy is Mary and Joe’s child.

Answer:

Either 1 and 3 or 2 and 3 form a set of premises. If 1 and 3 are premises, 2 may be deduced. If 2 and 3 are premises, 1 may be deduced. We say 1 and 2 are logically dependent. 3 is logically independent.

B. 1. x is an even integer.
   2. x + 1 is an odd integer.
   3. x + 2 is an even integer.

Answer:

All three statements are logically dependent. Given any one, the other two may be deduced. So each may serve as a premise with the other two as conclusions.

C. 1. Bob is taller than Jack.
   2. Jack is taller than Jim.
   3. Jim is taller than Gene.
   4. Gene is taller than Joe.

Answer:

All four statements are logically independent. All are premises — none can be deduced from the others. (There are, however, many conclusions which follow from these premises: Jim is taller than Joe, Jack is taller than Joe, Bob is taller than Joe, and so on.)

EXERCISES 24

Determine which of the statements may be taken as premises, and which as conclusions. Remember some sets may have several different premise-conclusion groupings.
A. 1. \( x + y = 2 \)
2. \( 2x - 4 = -2y \)
3. \( 3x - 6 = -3y \)

B. 1. Mr. \( x \) is Jim's father.
2. Jim is the son of Mr. \( x \).
3. Mrs. \( x \) is the wife of Mr. \( x \).
4. Mr. \( x \) is the husband of Mrs. \( x \).
5. Jane is Jim's sister.

C. 1. Bill is 5'9".
2. Bill is 6 inches shorter than Jim.
3. Jim is 6 inches taller than Bill.
4. Jim is 6'3".

D. 1. \( x \) is an even number.
2. \( x + 2 \) is an even number.
3. \( x + 4 \) is an even number.

---

**TASK 25**

Describe the ability to axiomatize and state its relation to making logical inferences.

**ANSWER 25**

Axiomatization involves determining a subset of logically independent statements from which other statements in a given set of statements can be deduced. The logically independent statements are called axioms (or premises). Axiomatization is in some sense the opposite of making logical inferences; the latter involves determining logical conclusions of a given set of axioms while the former involves determining axioms from a (larger) set of statements.

---

**TASK 26**

Illustrate what a person must do to show that he has the ability to axiomatize.

**ANSWER 26**

When presented with a set of statements, he must be able to identify a set of axioms for the set (of statements). He should also be able to tell if any two statements in the set are logically dependent or independent.
EXAMPLES 26

The following examples include both the tasks and possible correct answers.

A. Statements: 1. $x + 3 = y - 1$
   2. $x - y = -4$
   3. $2x + 8 = 2y$

   Answer: 1. $x - y = -4$ will serve as an axiom of set A.
   2. All the statements in set A are logically dependent.

B. Statements: 1. The flowers are ruined.
   2. The ground is wet.
   3. It is raining.
   4. If it is raining (then) the ground is wet.
   5. If the ground is wet (then) the flowers are ruined.

   Answer: Statements 3, 4, and 5 are axioms for set B. Modus Ponens is the deduction form.
   (4) "If it is raining the ground is wet" and (3) "It is raining" are axioms and independent. (2) "The ground is wet" is dependent on these. (5) "If the ground is wet the flowers are ruined" and (2) "The ground is wet" are independent and (1) "The flowers are ruined" is logically dependent on (5) and (2).

C. Statements: 1. All women are human beings.
   2. All human beings are mortal.
   3. Sue is mortal.
   4. All mortal beings die within 500 years.
   5. Sue is a woman.
   6. Sue will die within 500 years.
   7. All human beings die within 500 years.
   8. Sue is a human being.

   Answer: Statements 1, 2, 4 and 5 form a set of axioms. All other statements can be deduced from these using the syllogistic form.
   (7) follows from (2) and (4). (8) follows from (1) and (5).
   (6) follows from (7) and (8) (or (2), (4), (1) and (5)). Finally, (3) follows from (2) and (8) (or (2), (1) and (5)).

EXERCISES 26

A. 1. The cat is sick.
   2. The cat ate the meat.
   3. The cat is hungry.
   4. If the cat eats the meat it will get sick.
   5. If the cat is hungry it will eat the meat.
B. 1. All rattlesnakes are poisonous.
   2. All poisonous snakes must be avoided.
   3. This is a rattlesnake.
   4. This snake must be avoided.
   5. This snake is poisonous.

C. 1. x is an even number.
   2. y is an odd number.
   3. x + y is an odd number.
   4. x + x is an even number.
   5. y + y is an even number.
   6. x • y is an even number
   7. x • x is an even number.
   8. y • y is an odd number.

D. 1. x + y = 6
   2. x - 3 = y - 3
   3. x = 3
   4. 2x - 12 = -2y
SECTION 7. Heuristics and Combining Process Abilities

TASK 27

Given a problem situation and a worked out solution, identify the processing skills involved.

EXAMPLES 27

A. Find the sum of the first 50 odd positive integers.

1. \[1 = 1\]
   \[1 + 3 + 5 = 9\]
   \[1 + 7 = 4\]
   \[1 + 3 + 5 + 7 = 16\]

2. \[1 = 1\]
   \[1 + 3 = 4\]
   \[1 + 3 + 5 = 9\]
   \[1 + 3 + 5 + 7 = 16\]

3. \[1 = 1^2\]
   \[1 + 3 = 2^2\]
   \[1 + 3 + 5 = 3^2\]
   \[1 + 3 + 5 + 7 = 4^2\]

4. \[1 + 3 + 5 + ... + (2n - 1) = n^2\]

5. The sum of the first \(n\) odd integers is \(n^2\).

6. The sum of the first 50 odd integers is \(50^2 = 2500\)

Answer:

Processing skills used:
Step 1. Generate and solve some simpler problems of the same type (particu-
larizing).
Step 2. Reorder the display (detecting regularities).
Step 3. Recall the rule \(1 = 1^2, 4 = 2^2, 9 = 3^2, 16 = 4^2\), and rewrite the dis-
play (detecting regularities).
Step 4. Detect the regularity, \(1 + 3 + 5 + ... + (2n - 1) = n^2\).
Step 5. Translate from symbolic to verbal form.
Step 6. Particularize for the case, \(n = 50\).

B. Add the first 50 positive integers.
1. 1
   1 + 2
   1 + 2 + 3
   1 + 2 + 3 + 4
   1 + 2 + 3 + 4 + 5
   1 + 2 + 3 + 4 + 5 + 6
          = 1
          = 3
          = 6
          = 10
          = 15
          = 21

2. e.g., 1 + 2 + 3 + 4 + 5 + 6 = 21

3. The sum of corresponding pairs of terms is 2n + 1 (where n is the number of pairs).

4. (a) Number of Terms (b) Number of Pairs (c) Constant Sum of Pairs (d) Sum of All Terms
   
<table>
<thead>
<tr>
<th>Terms</th>
<th>Pairs</th>
<th>Constant Sum of Pairs</th>
<th>All Terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>7</td>
<td>21</td>
</tr>
</tbody>
</table>

5. The number of pairs is one-half the number of terms in the sum.
6. The constant sum of the pairs is one more than twice the number of pairs.
7. The sum of all terms is the product of the constant sum of pairs and the number of pairs.
8. The regularity in Step 5 tells us the number of pairs will be half the number of terms. Half of 50 is 25 pairs.
9. The regularity in Step 6 tells us that the constant sum of pairs will be 2\cdot25 + 1 or 51.
10. The regularity in Step 7 says the sum of all the pairs will be 25 \cdot 51 = 1275.

Answer:

Processing skills used:
Step 1. Try some simpler addition problems of the same form, i.e., particularize.
Step 2. Scan examples. Look for relationships among the terms i.e., simple ways of finding partial sums (detecting regularities).

Step 3. Detect a regularity for each of several number series with an even number (2n) of terms.

Step 4. Arrange data into a table.

Step 5. Detect regularity involving columns (a) and (b) of table.

Step 6. Detect regularity involving columns (b) and (c) of table.

Step 7. Detect regularity involving columns (b), (c), and (d) of table.

Step 8. Particularize regularity of Step 5.


EXERCISES 27

A. Identify the processing skills involved in Example A without referring to the answer.

B. The following are steps that might be used to determine the sum of the first 101 positive integers. Identify the processing skills involved at each step.

Step 1.

\[
\begin{align*}
1 & \quad = 1 \\
1 + 2 & \quad = 3 \\
1 + 2 + 3 & \quad = 6 \\
1 + 2 + 3 + 4 & \quad = 10 \\
1 + 2 + 3 + 4 + 5 & \quad = 15 \\
1 + 2 + 3 + 4 + 5 + 6 & \quad = 21 \\
1 + 2 + 3 + 4 + 5 + 6 + 7 & \quad = 28
\end{align*}
\]

Step 2. e.g., \(1 + 2 + 3 + 4 + 5 + 6 + 7 = 28\)

\[
\begin{align*}
1 + 2 + 3 + 4 + 5 & \quad = 15 \\
& \quad = 7 \\
& \quad = 6
\end{align*}
\]

Step 3.

<table>
<thead>
<tr>
<th>Number of terms (n)</th>
<th>Number of pairs</th>
<th>Constant Sum of Pairs</th>
<th>Sum of All Terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>7</td>
<td>15</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>9</td>
<td>28</td>
</tr>
</tbody>
</table>

37
Step 4. The number of pairs is \((n-1)/2\).
Step 5. The constant sum of pairs is \(n + 2\).
Step 6. The sum of all terms is \[((n-1)/2)(n+2)\] + 1.
Step 7. The number of pairs is \((101-1)/2 = 100/2 = 50\).
Step 8. The constant sum of pairs is \(101 + 2 = 103\).
Step 9. The sum of all terms is \(50 \times 103 + 1 = 5151\).

TASK 28

Describe three general heuristics or search procedures for solving problems.

ANSWER 28

A. One form of the translation heuristic involves interpreting a verbal statement of a problem and then representing or describing its meaning as a set of equations.

B. A second type of translation heuristic involves translating a given concrete display into symbolic form while preserving the original regularities. In doing this, we must first decide what aspects of the display to preserve by abstracting certain easy to detect regularities, then describe what has been abstracted in symbolic form. It is often easier to detect additional regularities in the symbolic display than in the concrete one.

C. The particularization heuristic may be used to help solve a complex general problem by first considering a simpler problem of the same type. The solution of the simpler problem may offer a key to the solution of the complex problem. The first step involves focusing on certain aspects of the given problem and particularizing on the basis of these aspects.

TASK 29

Solve problems using the heuristics stated and discussed in Answer 28.

EXAMPLES 29

A. Translate the following problem statement into a set of equations (and solve).

John can paint a room in six hours, and his older brother Joe can paint the room in three hours. How long will it take the two brothers working together to paint the room?

Answer:

Interpreting the verbal description: John works at the fixed rate of \(1/6\) job per hour. Joe works at the rate of \(1/3\) job per hour. Their combined rate (when they both work together) is \((1/6 + 1/3)\) job per hour. Describing the meaning in equation form:
Chapter 1
Section 7

Rate x time = total job

\((1/6 + 1/3) \times ? = 1\)

Solving we get \((1/6 + 2/6) \times ? = 1 \text{ or } ?/2 = 1 \text{ or } ? = 2 \text{ hours. Thus, the time required for the job is two hours.}\)

B. Translate a concrete (iconic) display to a symbolic display so that they exhibit the same regularities.

1. Is the operation of composing rotations of a square commutative?
2. Is there an identity element in the set of rotations?

Concrete Display:

First, visualize the regularities among the rotations of a square that must be preserved in the symbolic representation. To help, use icons.

\[
\begin{align*}
\begin{array}{c}
\text{A rotation thru } 0^\circ. \\
\text{A rotation thru } 90^\circ. \\
\text{A rotation thru } 180^\circ. \\
\text{A rotation thru } 270^\circ.
\end{array}
\end{align*}
\]

Answer:

Let 0, 1, 2, and 3 represent the four basic rotations of 0°, 90°, 180°, and 270°, respectively. Now consider the composition of pairs of these rotations e.g., a rotation of 90° followed by a rotation of 270° is the same as a rotation of 360° or no rotation at all (0°), or, 1 + 3 = 0. Similarly, 1 + 2 = 3, 3 + 3 = 2, 0 + 2 = 2, 0 + 0 = 0, etc. Next, represent the composition of rotations in a table.

\[
\begin{array}{c|cccc}
+ & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 1 & 2 & 3 \\
1 & 1 & 2 & 3 & 0 \\
2 & 2 & 3 & 0 & 1 \\
3 & 3 & 0 & 1 & 2 \\
\end{array}
\]

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Chapter 1
Section 7

[Note: The element at the intersection of a row and a column is the composition of the element at the left end of the row and the element at the top of the column.] Now, the two original questions can be answered. 1. Because the array is symmetric with respect to the "upper left to lower right," diagonal the composition of rotations of a square is commutative: $2 + 3 = 3 + 2$, $2 + 1 = 1 + 2$, etc. 2. Since 0 composed with any rotation gives that rotation again, we say 0 is the identity rotation: $0 + 0 = 0$, $0 + 1 = 1$, $0 + 2 = 2$, etc.

C. Given a set with n elements, how many subsets does it have? [Note: We have already found the solution to this question; however, here we will use a slightly different approach to illustrate the heuristic of particularization.]

Answer:

In order to solve this general problem, we first particularize and construct some specific (and simpler) problems of the same type. Specifically, let $A = \{1, 2, 3\}$, be a set of 3 elements, and $A' = \{1, 2\}$ be a set with 2 elements. $A$ has the following 8 subsets:

$\{1, 2, 3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\}$.

$A'$ has the following 4 subsets:

$\{1, 2\}$, $\{1\}$, $\{2\}$, $\{\}$.

Notice also that a set with 1 element has 2 subsets, the empty set and itself. The empty set, of course, has just 1 subset, itself. Performing the same analysis with other particular sets and summarizing the results in a table we get:

<table>
<thead>
<tr>
<th>No. of Elements</th>
<th>No. of Subsets</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
</tr>
</tbody>
</table>

From this table, it is easy to detect the regularity: A set of $n$ elements has $2^n$ subsets. This result may be verified with sets having different numbers of elements. Proving that the regularity works for all sets, however, would require a more involved argument.

EXERCISES 29

A. Jane can clean the house in 8 hours working by herself, and Jane and her younger sister, Sue, can clean the same house in 6 hours by working together. How long would it take Sue to clean the house working alone? (Hint: Let Sue's rate be $1/x$ job per hour.)
B. Characterize composition of the rotations of a regular hexagon in a symbolic array (as in Example B above). Is there an identity element for the array? Is it commutative?

C. Consider a set $A$ with four elements and a subset $A'$ of it with three elements. Use this, and possibly other, pairs of sets to determine the relationship between the number of subsets of a set with $n$ elements and the number of subsets of a set with $n - 1$ elements (i.e., find the relationship which holds for all $n$).
PART 2

BASIC IDEAS IN MATHEMATICS AND LOGIC

CHAPTER 2

SETS, RELATIONS, AND OPERATIONS

SECTION 1. The Nature of the Real World: Things, Relations, and Operations

TASK 1

Classify real world entities as "things," "relations," or "operations."

RULE 1

A. If an entity is physical, meaningful in itself, and can be considered without reference to anything else, then it is called a "thing."
B. If an entity draws its meaning from two or more things, then it is called a "relation."
C. If an entity is an action on things or relations in the real world, then it is called an "operation."

(*Some entities may be complex and involve combinations of things, relations, and/or operations.)

EXAMPLES 1

A. Things: football, desk, set, room.
B. Relations: is the uncle of, is the leader of, is in debt to, is a division of, is equal to.
C. Operations: scoring, adding, getting dressed, fighting, dividing.
EXERCISES 1

A. pillow
B. destroying
C. is the end of
D. horse
E. canoeing
F. towing
G. cart
H. is a multiple of
I. is larger than
SECTION 2. Sets and Subsets

TASK 2

Given a finite universe of elements and the defining property of a set in the universe, represent that set.

RULE 2

List the elements of the universe satisfying the defining property of the set and enclose this list in brackets "[ ]".

EXAMPLES 2

A. Universe: The integers between 1 and 21.
   Defining Property: Integers (between 1 and 21) divisible by 2.
   Set: \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}

B. Universe: The Presidents of the United States between 1900 and 1970.
   Defining Property: The Presidents of the United States (between 1900 and 1970) who died in office
   Set: \{McKinley, F. D. Roosevelt, J. F. Kennedy, Harding\}.

EXERCISES 2

A. Universe: The integers between 1 and 21.
   Defining Property: Integers (between 1 and 21) divisible by 5.

B. Universe: The letters of the English alphabet.
   Defining Property: Letters (of English alphabet) called vowels.

C. Universe: The multiples of 2 between 1 and 51.
   Defining Property: Multiples of 4 (between 1 and 51).

D. Universe: The Presidents of the United States after 1900.
   Defining Property: Female Presidents of United States after 1900.

TASK 3

Represent a given set using set builder notation.

RULE 3

Find a property, \(P\), which defines the set and then write \(\{x | P(x)\}\) which is read "The set of all elements, \(x\), such that \(P(x)\)".

EXAMPLES 3

A. Given: \[2, 4, 6, 8, \ldots, 20\]
   Write: \(\{x | x \text{ is an even integer between 1 and } 21\}\).
   Read: "The set of all elements \(x\) such that \(x\) is an even integer between 1 and 21."
B. Given: \{McKinley, Harding, F. D. Roosevelt, J. F. Kennedy\}.
Write: \{x | x is a President of the United States between 1900 and 1970 who died in office\}
Read: "The set of all elements, x, such that x is a President of the United States between 1900 and 1970 who died in office."

EXERCISES 3

A. \{5, 10, 15, ..., 35\}
B. \{do, re, me, fa, so, la, te\}
C. \{hydrogen, oxygen\}
D. \{Alaska, Washington, Oregon, California, Hawaii\}

TASK 4

Given two sets, A and B, determine whether or not A is a subset of B.

RULE 4

A. If set A has a small finite number of elements and every element of A is an element of B, then A is a subset of B. (Write \(A \subseteq B\)). Otherwise, A is not a subset of B.

B. If A has a large number of elements consider the defining property of each set. If every element that satisfies the defining property of A (i.e., is an element of A), also satisfies the defining property of B, then A is a subset of B. (Write \(A \subseteq B\)). Otherwise, A is not a subset of B.

EXAMPLES 4

A. Set A: \{2, 4, 6, 8, 10\}. Set B: \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}
Answer:
Since each element in Set A is also in set B, A is a subset of B. \(A \subseteq B\).

B. Set A: \{x | x is a multiple of 2\}. Set B: \{x | x is a multiple of 3\}.
Answer:
Since an element of set A, e.g., 4, is not a multiple of 3 (i.e., is not in set B), A is not a subset of B.

EXERCISES 4

A. Set A: \{1, 3, 5, 7, 9\}. Set B: \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.
B. Set A: \{1, 2, 3, 4, 5\}. Set B: \{2, 3, 4, 5, 6\}.
TASK 5

Given a set, A, and the finite universe to which A belongs, determine the complement of A.

RULE 5

List all elements in the universe not belonging to set A and enclose the list in brackets. The defining property of the complement of A is: "an element in the universe which does not belong to Set A (which does not satisfy the defining property of set A.)"

EXAMPLES 5

A. Set A: \{1, 3, 5, 7, 9\}
   Universe: \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}
   Complement of A: \{0, 2, 4, 6, 8, 10\}

B. Set A: \{Edmund, Teddy, Hubert\}
   Universe: \{Edmund, Richard, Spiro, Teddy, Strom, Hugh, Hubert\}
   Complement of A: \{Richard, Spiro, Strom, Hugh\}

EXERCISES 5

A. Set A: \{5, 10, 15, 20\}
   Universe: \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}

B. Set A: \{1, 3, 7, 5, 9\}
   Universe: \{1, 2, 3, 5, 9, 8, 7\}

C. Set A: \{January, March, May, July, August, October, December\}
   Universe: \{January, February, March, April, May, June, July, August, September, October, November, December\}

D. Set A: \{□, *, ○, △\}
   Universe: \{*, ○, △, □\}
SECTION 3. Correspondence

TASK 6

Given two finite sets A and B and a defined correspondence between them, represent the correspondence by drawing an arrow, ←→, between each pair of corresponding elements.

RULE 6

List the elements of A and B in respective parallel columns. Using the defined correspondence draw an arrow from each element of A to the corresponding element (or elements) in B. If there are elements of A which do not correspond to any element of B just list the elements (without the correspondence arrow).

EXAMPLES 6

A. Set A: \{1, 2, 3, 4, 5\}
   Set B: \{2, 4, 6, 8, 10\}
   Correspondence: Pair each element of A with its double in B.
   Answer: \[
   \begin{array}{ll}
   \text{Set A} & \text{Set B} \\
   1 & \leftrightarrow 2 \\
   2 & \leftrightarrow 4 \\
   3 & \leftrightarrow 6 \\
   4 & \leftrightarrow 8 \\
   5 & \leftrightarrow 10 \\
   \end{array}
   \]

B. Set A: \{Nixon, Johnson, Kennedy, Eisenhower\}
   Correspondence: Pair each President with a year he was in office.
   Answer: \[
   \begin{array}{ll}
   \text{Set A} & \text{Set B} \\
   Nixon & \uparrow 1971 \\
   & \uparrow 1969 \\
   Johnson & \uparrow 1967 \\
   & \uparrow 1965 \\
   Kennedy & \uparrow 1962 \\
   Eisenhower & \uparrow 1959 \\
   & \uparrow 1957 \\
   & \uparrow 1951 \\
   \end{array}
   \]

EXERCISES 6

A. Set A: \{1, 2, 3, 4, 5\}
   Set B: \{5, 10, 15, 20, 25\}
   Correspondence: Pair each element in A with the number that is five times its value in B.
B. Set A: \([-3, -2, -1, 0, 1, 2, 3] \) 
Set B: \([0, 1, 4, 9] \)  
Correspondence: Pair each element in A with its square in B.

C. Set A: \([x | x \text{ is a month in the year}] \)  
Set B: \([28, 29, 30, 31] \)  
Correspondence: Pair each element in A with its number of days in B.

D. Set A: \([y | y \text{ is a day of the week}] \)  
Set B: \([1, 2, 3, 4, 5, 6, 7, 8, 9, 10] \)  
Correspondence: Pair each element in A with the number of letters in its spelling in B.

TASK 7

Given two finite sets A and B and a correspondence between them, determine if the correspondence is one-to-one.

RULE 7

Apply Rule 6. If (1) each element of A which corresponds to some element of B corresponds with only one element of B and (2) each element of B which corresponds to some element of A, corresponds to only one element of A, then the correspondence is one-to-one; otherwise it is not.

EXAMPLES 7

A. Set A: \([1, 2, 3, 4] \) 
Set B: \([2, 4, 6, 8, 10] \)  
Correspondence: Pair each element in A with its double in B.

Answer:

By Rule 6:  
\[
\begin{array}{ccc}
A & & B \\
1 & \leftrightarrow & 2 \\
2 & \leftrightarrow & 4 \\
3 & \leftrightarrow & 6 \\
4 & \leftrightarrow & 8 \\
& & 10 \\
\end{array}
\]

Because each element of A and B in the correspondence corresponds to only one element of B and A, respectively, the correspondence is one-to-one.

B. Set A: \([-2, -1, 0, 1, 2] \)  
Set B: \([0, 1, 4] \)  
Correspondence: Pair each element in A with its square in B.
Answer:

By Rule 6:

```
<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>4</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>
```

Because the element 4 in B is paired with two different elements in A, the correspondence is not one-to-one.

EXERCISES 7

A. Set A: \{x/x is a state of the United States bordering Canada\}
   Set B: \{y/y is a capitol of a state of the United States bordering Canada\}
   Correspondence: Each state is paired with its capitol.

B. Set A: \{do, re, me, fa, so, la, te\}
   Set B: \{1, 2, 3, 4, 5, 6, 7, 8\}
   Correspondence: Pair the first element of Set A with the first element of Set B, the second with the second, etc.

C. Set A: \{-3, -2, -1, 0\}
   Set B: \{0, 1, 2, 3\}
   Correspondence: Each element in A(B) is paired with its negative in B(A).

D. Set A: \{1, 2, 3, 4, 5\}
   Set B: \{7, 14, 21, 28, 35, 42\}
   Correspondence: Each element in A is paired with its product by 7 in B.

TASK 8

Given two finite sets A and B, construct a one-to-one correspondence between them.

RULE 8

List all the elements of sets A and B in respective parallel columns. Starting from the top of the A column, draw an arrow from each element in A to an element in B, making sure that no element of B has more than one arrow to it. Continue until either there are no more elements in set A or every element in B has an arrow to it.

EXAMPLES 8

A. Set A: \{a, e, i, o, u\}
   Set B: \{1, 2, 3, 4, 5\}

Answer:

A possible one-to-one correspondence:
A possible one-to-one correspondence:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>□</td>
</tr>
<tr>
<td>e</td>
<td>△</td>
</tr>
<tr>
<td>i</td>
<td>○</td>
</tr>
<tr>
<td>o</td>
<td>▽</td>
</tr>
<tr>
<td>u</td>
<td>□</td>
</tr>
</tbody>
</table>

EXERCISES 8

A. Set A: \{-1, -2, -3, -4, -5\}
   Set B: \{2, 4, 6, 8, 10\}
B. Set A: \{John, Mike, Jack, James\}
   Set B: \{Ann, Kate, Mary, Therese, Julie\}
C. Set A: \{x | x is an even integer between 1 and 11\}
   Set B: \{y | y is an odd integer between 12 and 22\}
D. Set A: \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}
   Set B: \{I, II, III, IV, V, VI, VII, VIII, IX, X\}

TASK 9

Given two finite sets A and B, determine if they have the same number of elements.

RULE 9

Apply Rule 8. If each element in A corresponds to an element in B, and each element in B corresponds to an element in A, then the sets have the same number of elements; otherwise they do not.

EXAMPLES 9

A. Set A: \{a, e, i, o, u\}
   Set B: \{1, 2, 3, 4, 5\}
Answer:

Applying Rule 8, get a 1-1 correspondence:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>1</td>
</tr>
<tr>
<td>e</td>
<td>2</td>
</tr>
<tr>
<td>i</td>
<td>3</td>
</tr>
<tr>
<td>o</td>
<td>4</td>
</tr>
<tr>
<td>u</td>
<td>5</td>
</tr>
</tbody>
</table>

Because each element of A is paired with an element of B, and vice versa, the two sets have the same number of elements.

B. Set A: \( \{ x \mid x \text{ is an odd number between 2 and 12} \} \)
   Set B: \( \{ y \mid y \text{ is an even number between 13 and 25} \} \)

Answer:

Applying Rule 8 get a 1-1 correspondence:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>14</td>
</tr>
<tr>
<td>5</td>
<td>16</td>
</tr>
<tr>
<td>7</td>
<td>18</td>
</tr>
<tr>
<td>9</td>
<td>20</td>
</tr>
<tr>
<td>11</td>
<td>22</td>
</tr>
<tr>
<td></td>
<td>24</td>
</tr>
</tbody>
</table>

Because there is no element in A paired with 24 in B, the two sets do not have the same number of elements.

EXERCISES 9

A. Set A: \( \{ \bigcirc, \Delta, \Box, \square \} \)
   Set B: \( \{ 4, 3, 2, 1 \} \)
B. Set A: \( \{ 1, 2, 3, 4, 5, 6, 7 \} \)
   Set B: \( \{ 1, 4, 9, 16, 25, 36, 49, 64 \} \)
C. Set A: \( \{ x \mid x \text{ is a day of the week} \} \)
   Set B: \( \{ \text{do, re, me, fa, so, la, te} \} \)
D. Set A: \( \{ x \mid x \text{ is a letter in the English alphabet} \} \)
   Set B: \( \{-1, -2, -3, -4, \ldots, -26 \} \)

TASK 10

Given two finite sets A and B and a correspondence defined between them, determine if the correspondence is many-to-one.
RULE 10

Apply Rule 6. If at least one element of Set B corresponds to more than one element of Set A (and no element of Set A corresponds to more than one element of B), then the correspondence is many-to-one; otherwise it is not.

EXAMPLES 10

A. Set A: \{-3, -2, -1, 0, 1, 2, 3\}
Set B: \{0, 1, 4, 9\}
Correspondence: Each element in A is paired with its square in B.

Answer:

Applying Rule 6:

Because 1, 4, and 9 of Set B each correspond to more than one element of Set A (and no element of Set A corresponds to more than one element of B), the correspondence is many-to-one.

B. Set A: \{x | x is a state of the United States bordering the Pacific Ocean\}
Set B: \{y | y is a capital city of a state bordering the Pacific Ocean\}
Correspondence: Each city is paired with its state.

Answer:

Applying Rule 6

Because no element of Set B corresponds to more than one element of Set A, the correspondence is not many-to-one. (It is one-to-one)

EXERCISES 10

A. Set A: \{1, 2, 3, 4, 5, 6, 7, 8\}
Set B: \{10, 8, 6, 4, 2, 0, -2, -4\}
Correspondence: Pair each number in A with its double in B.

B. Set A: \{Jets, Mets, Knickerbockers, Rams, Dodgers\}
Set B: \{New York, Los Angeles\}
Correspondence: Pair each athletic team with the city it represents.

C. Set A: \{16, 9, 4, 1, 0\}
Set B: \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}
Correspondence: Pair each number in A with its square root in B.

D. Set A: \{x \mid x \text{ is a sock that Tom, Jim, or Fred is wearing}\}
Set B: \{Tom, Jim, Fred\}
Correspondence: Pair each sock with the person wearing it.

---

**TASK 11**

Given two finite sets A and B and a correspondence defined between them, determine if the correspondence is one-to-many.

**RULE 11**

Apply Rule 6. If at least one element of Set A corresponds to more than one element of Set B, then the correspondence is one-to-many; otherwise it is not.

**EXAMPLES 11**

A. Set A: \{2, 3, 7\}
Set B: \{4, 8, 9, 15, 35\}
Correspondence: Pair each element of Set A with each of its multiples in Set B.

Answer:

Applying Rule 6:

```
  A  B
 2  →  4
 3  →  9
 7  → 15
```

Because 2 and 3 of Set A correspond to more than one element of set B, the correspondence is one-to-many.

B. Set A: \{x \mid x \text{ is an even number between 1 and 11}\}
Set B: \{y \mid y \text{ is an odd number between 12 and 22}\}
Correspondence: If x is in A pair it with \(x + 11\) in B.

Answer:

Applying Rule 6:
Because no element of Set A corresponds to more than one element of set B, the correspondence is not one-to-many.

### EXERCISES 11

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>13</td>
</tr>
<tr>
<td>4</td>
<td>15</td>
</tr>
<tr>
<td>6</td>
<td>17</td>
</tr>
<tr>
<td>8</td>
<td>19</td>
</tr>
<tr>
<td>10</td>
<td>21</td>
</tr>
</tbody>
</table>

### A. Set A: \[9, 11, 13, 17, 29\]
Set B: \[22, 18, 34, 26, 58\]
Correspondence: Pair each number in A with its double in B.

### B. Set A: \[81, 64, 49\]
Set B: \[-7, -8, -9, 7, 8, 9\]
Correspondence: Pair each number in A with its square root in B.

### C. Set A: \{Ford, Chrysler, General Motors\}
Set B: \{Falcon, Impala, Nova, Cadillac, Dodge\}
Correspondence: Pair each manufacturer with the car model(s) it makes.

### D. Set A: \{American Football Conference, National Football Conference\}
Set B: \{Colts, Jets, Rams, 49ers, Vikings\}
Correspondence: Pair each football team with its conference.

### TASK 12

Given a correspondence (relation) between two finite sets A and B, determine if it is a function.

### RULE 12

Apply Rule 6. If the correspondence satisfies Rule 7 or Rule 10, then it is a function; otherwise it is not.

### EXAMPLES 12

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>y</td>
</tr>
</tbody>
</table>

### A. Set A: \{a, e, i, o, u\}
Set B: \{1, 2, 3, 4, 5\}
Correspondence: Pair the first element in A with the first element in B, the second with the second, etc.

Answer:

Because this correspondence satisfies Rule 7, it is a function.

### B. Set A: \{x | x is a United States Senator\}
Set B: \{y | y is a state of the United States\}
Correspondence: Each Senator is paired with the state he (she) represents.

Answer:

Because this correspondence satisfies Rule 10, it is a function.

EXERCISES 12

A. Set A: \( \{ x \mid x \text{ is a letter of the alphabet} \} \)
   Set B: \( \{ y \mid y \text{ is an even number between 10 and 70} \} \)
   Correspondence: Same as in Example 12A.

B. Set A: \( \{ 0, 1, 4, 9, 16 \} \)
   Set B: \( \{ -4, -3, -2, -1, 0, 1, 2, 3, 4 \} \)
   Correspondence: Pair each element in A with its square root in B.

C. Set A: \( \{ x \mid x \text{ is a New England state of the United States} \} \)
   Set B: \( \{ y \mid y \text{ is a United States Senator from a New England state} \} \)
   Correspondence: Pair each state with each of its Senators.

D. Set A: \( \{ -3, -2, -1, 0, 1, 2, 3 \} \)
   Set B: \( \{ 0, 1, 4, 9 \} \)
   Correspondence: Pair each number in A with its square in B.

TASK 13

Given a correspondence between Sets A and B represented iconically by arrows (i.e., \( a_i \leftrightarrow b_j \)), represent the correspondence symbolically using ordered pairs.

RULE 13

For each \( a_i \leftrightarrow b_j \), write \( (a_i, b_j) \).

EXAMPLES 13

A. Given: \( 1 \leftrightarrow 2 \)
   Answer: \( (1, 2) \)
   \( 2 \leftrightarrow 4 \)
   \( (2, 4) \)
   \( 3 \leftrightarrow 6 \)
   \( (3, 6) \)
   \( 4 \leftrightarrow 8 \)
   \( (4, 8) \)

B. Given: January \( \rightarrow \) February \( \rightarrow \) March \( \rightarrow \) April \( \rightarrow \) May \( \rightarrow \) June \( \rightarrow \) July
   Answer: (January, 31)
   (February, 28)
   (February, 29)
   (March, 31)
   (April, 30)
   (May, 31)
   (June, 30)
   (July, 31)
EXERCISES 13

A. 1
   -1 ←→ 1
   2  ←→ 4

B. I ←→ 1
   V ←→ 5
   X ←→ 10
   L ←→ 50
   C ←→ 100

TASK 14

Given a correspondence between two sets represented iconically, state a verbal rule telling how the elements are paired.

RULE 14

Try to detect a regularity (See Chapter 1) among the given pairs, \( a_i \leftrightarrow b_i \), of elements. Then state this regularity in the form "Each \( a_i \) corresponds to \( f(a_i) = b_i \)" where \( f \) is a rule for generating \( b_i \) from \( a_i \) for all \( a_i \).

EXAMPLES 14

A. Given: 1 ←→ 1
   2 ←→ 4
   3 ←→ 9

Answer:

"Each number 1, 2, 3 corresponds to its square."

B. Given: a ←→ 1
   b ←→ 2

Answer:

"Each letter corresponds to the number which represents its position in the alphabet."

EXERCISES 14

A. 1 ←→ I
   2 ←→ II
   5 ←→ V
   10 ←→ X
SECTION 4. *Relations as States and Functions as Operators*

**TASK 15**

Given a relation classify it as unary, binary, ternary, ..., n-ary.

**RULE 15**

If one element or variable is considered, call the relation unary. If two elements or variables are considered, call the relation binary. If three elements or variables are considered, call the relation ternary. If n elements or variables are considered, call the relation n-ary.

**EXAMPLES 15**

A. Given; the relation of identity which each element has to itself.

Answer:

This can be considered unary since one element is considered.

B. Given: the relation "less than (<)"

Answer:

Less than is considered binary since two numbers can be related to each other by less than.

C. Given: the relation of "three numbers and their average."

Answer:

This is four-ary, e.g., (2, 3, 4, 3), (-6, -5, 11, 0), since four numbers are related.

**EXERCISES 15**

A. The relation "greater than (>)"
B. The relation "unequal to (#)"
C. The relation of "four numbers and their average."
D. The relation of "two numbers and their sum."

**TASK 16**

Given an operation, classify it as unary, binary, ternary, ..., n-ary.
RULE 16

If one element or variable is operated on, call the operation unary. If two elements or variables are operated on, call the operation binary. If three elements or variables are operated on, call the operation ternary. If n elements or variables are operated on, call the operation n-ary.

EXAMPLES 16

A. Given: the operation of "squaring" a number.
Answer:
This is unary because one number is operated on to produce a unique second number.

B. Given: the operation "addition (+)."
Answer:
This is binary because two numbers are operated on to produce a unique third number.

C. Given: the operation "finding the volume of a room."
Answer:
This is ternary because three numbers are operated on (length, width, and height) to produce a unique fourth number (the volume).

EXERCISES 16

A. The operation "multiplication."
B. The operation "finding the square root."
C. The operation "averaging six numbers."
D. The operation "finding the median of 19 test scores."
SECTION 5. Equivalence

TASK 17

Given two entities and a relation between them, determine whether or not they are equivalent according to the relation.

RULE 17

If the two entities satisfy the given relation then the entities are equivalent; otherwise they are not equivalent.

EXAMPLES 17

A. Entities: a dollar bill and four quarters.
   Relationship: "has the same monetary value as."

   Answer:

   Because four quarters and a dollar bill have the same monetary value, they are equivalent.

B. Entities: a dollar bill and four quarters.
   Relationship: "has the same buying value in a coin vending machine as."

   Answer:

   Because a dollar bill cannot be used in a vending machine for coins only, the two entities are not equivalent.

EXERCISES 17

A. Entities: a bicycle and a car.
   Relationship: "have same number of wheels as!"

B. Entities: a bicycle and a car.
   Relationship: "both can convey a single person over a short distance."

C. Entities: a yard stick and a 12" piece of wood.
   Relationship: "is the same length as."

D. Entities: a yard stick and a 12" piece of wood.
   Relationship: "both can be used to measure a two foot long piece of wood."

TASK 18

Given a set of subsets of a finite universe, determine whether or not these subsets exhaust the universe.
RULE 18

If each element of the universe is in (at least) one of the subsets, then the subsets exhaust the universe; otherwise they do not.

EXAMPLES 18

A. Sets of subsets: \([\{1, 3, 5, 7, 9\}, \{2, 4, 6, 8, 10\}]\)
   Universe: \([1, 2, 3, 4, 5, 6, 7, 8, 9, 10]\)
   Answer:
   Because each element in the universe is in one of the subsets, the subsets exhaust the universe.

B. Set of subsets: \([\{a\}, \{a, b\}, \{a, b, c\}, \{b, c\}]\)
   Universe: \([a, b, c, d]\)
   Answer:
   Because element d is in none of the subsets, the subsets do not exhaust the universe.

EXERCISES 18

A. Set of subsets: \([\{\emptyset\}, \{\square\}, \{\triangle\}]\)
   Universe: \([\emptyset, \square, \triangle]\)

B. Set of subsets: \([\{x\mid x \text{ is a month of the year having 30 days}\}, \{y\mid y \text{ is a month of the year having 31 days}\}]\)
   Universe: \([x\mid x \text{ is a month of the year}\]

C. Set of subsets: \([\{1, 2\}, \{1, 2, 3\}, \{2, 3, 4\}, \{3, 4\}]\)
   Universe: \([1, 2, 3, 4, 5]\)

D. Set of subsets: \([\{a, b, c, d, e\}]\)
   Universe: \([a, b, c, d, e]\)

---

TASK 19

Given two finite sets A and B, determine whether or not they are disjoint.

RULE 19

If no element of set A is an element of set B, then the sets are disjoint; otherwise they overlap (i.e., are not disjoint).

EXAMPLES 19

A. Set A: \([2, 4, 6, 8, 10]\)
   Set B: \([1, 3, 5, 7, 9]\)
Answer:

Because no element of set A is an element of set B, the sets are disjoint.

B. Set A: \{1, 2, 3\}
   Set B: \{3, 4, 5\}

Answer:

Because 3 is an element of both sets, the sets are not disjoint.

EXERCISES 19

A. Set A: \{x|x is an even integer between 10 and 20\}
   Set B: \{y|y is a multiple of 3 between 4 and 25\}

B. Set A: \{x|x is a United States Senator\}
   Set B: \{y|y is a member of the House of Representatives\}

C. Set A: \{x|x is a state of the United States that borders on the Atlantic Ocean\}
   Set B: \{y|y is one of the original 13 states\}

D. Set A: \{□, △, ○, ▽\}
   Set B: \{x|x is a figure with 5 sides\}

TASK 20

Given a set of subsets of a finite universe, determine whether or not this set (of subsets) is a partition. If it is a partition, then the sets are equivalence classes.

RULE 20

If the subsets exhaust the universe (Rule 18) and if each pair of subsets is disjoint (Rule 19), then the set of subsets is a partition; otherwise it is not.

EXAMPLES 20

A. Universe: \{1, 2, 3, 4, 5, 6\}
   Set of Subsets: \{[1, 3, 5], [2, 4, 6]\}

Answer:

Because the set of subsets exhausts the universe and the two subsets are disjoint, the set is a partition (i.e., is a set of equivalence classes).

B. Universe: \{□, ○, □, △, ▽\}
   Set of Subsets: \{[□, ○, □], [△, ▽, □]\}
Answer:

Because □ is in both subsets, Rule 19 is not satisfied and the subsets do not form a partition (i.e., are not equivalence classes).

EXERCISES 20

A. Universe: {□, □, □, □, □, □, □, □}
   Set of subsets: {[□, □, □, □], [□, □, □, □]}
B. Universe: {1, 2, 3, 4, 5, 6, 7, 8, 9}
   Set of subsets: {[1, 2, 3], [4, 5, 6], [7, 8], [9]}
C. Universe: {x | x is an integer between 1 and 50}
   Set of subsets: {[y | y is an even integer between 1 and 50], [z | z is an odd integer between 1 and 50]}
D. Universe: {11, 12, 13, 14, 15, 16, 17}
   Set of subsets: {[11, 13, 17], [12, 14], [16]}

TASK 21

Given a finite set A and a relation R defined on the set, determine whether or not R is reflexive on A.

RULE 21

If every element x of set A is related to itself by the relation R (x R x), then R is reflexive on A; otherwise it is not.

EXAMPLES 21

A. Set A: {1, 2, 3}
   Relation R: equality (=)

   Answer:

   Because 1 = 1, 2 = 2, and 3 = 3, equality is reflexive on {1, 2, 3}.

B. Set A: {1, 2, 3}
   Relation R: less than (<)

   Answer:

   Because 1 < 1 is false, less than is not reflexive on {1, 2, 3}.

EXERCISES 21

A. Set A: {9, 7, 13, 27}
   Relation R: equality (=)
Given a finite set $A$ and a relation $R$ defined on the set, determine whether or not $R$ is symmetric on $A$.

**RULE 22**

$R$ is symmetric on set $A$ if for every pair of elements $x$ and $y$, of $A$, $yRx$ whenever $xRy$; otherwise $R$ is not symmetric on $A$.

**EXAMPLES 22**

A. Set $A$: \{1, 2, 3\}
Relation $R$: less than ($<$)

Answer:

Because $1 < 2$ but not $2 < 1$, the relation, less than, is not symmetric on \{1, 2, 3\}.

B. Set $A$: \{\# , \# , \# , \# , \# \}
Relation $R$: is the same shape as.

Answer:

Because \# , \# , and \# all have the same shape, if one is the same shape as a second, then the second must have the same shape as the first. This property also holds for \# and \# . None of the other pairs satisfy $R$. Hence, $R$ is symmetric on $A$.

**EXERCISES 22**

A. Set $A$: \{1, 4, 3/3, 12/3\}
Relation $R$: equality ($=$)

B. Set $A$: \{\# , \# , \# , \# , \# , \# , \# , \# \}
Relation $R$: has the same number of sides as

C. Set $A$: \{16/4, 2, 4/2, 4, 9/3\}
Relation $R$: inequality ($\neq$)

D. Set $A$: \{9, 7, 16, 23, 27\}
Relation $R$: greater than ($>$)
TASK 23

Given a finite set A and a relation R, defined on the set, determine whether or not R is transitive on A.

RULE 23

R is transitive if for every triple of elements x, y, and z of set A, x R z, whenever x R y and y R z. Otherwise, R is not transitive.

EXAMPLES 23

A. Set A: \{1, 2, 3\}
   Relation R: less than ( < )
   Answer:
   Because 1 < 2, 2 < 3, and 1 < 3, the relation less than is transitive.
B. Set A: \{\textbullet, \circ, \blacksquare, \lozenge, \square\}
   Relation R: is a different shape than
   Answer:
   Because \circ is a different shape than \blacksquare, and \blacksquare is a different shape than \lozenge but \textbullet is not a different shape than \circ, the relation "is a different shape than" is not transitive.

EXERCISES 23

A. Set A: \{1/1, 2/2, 3/3, 3/4, 6/8, 9/12\}
   Relation R: equality ( = )
B. Set A: \{1, 2, 3, 4/2, 3/3\}
   Relation R: inequality ( ≠ )
C. Set A: \{\blacksquare, \lozenge, \lozenge, \blacksquare, \lozenge, \circ, \blacklozenge\}
   Relation R: has the same number of sides as
D. Set A: \{\blacksquare, \lozenge, \lozenge, \blacksquare, \lozenge, \circ, \blacklozenge\}
   Relation R: has a different number of sides than

TASK 24

Given a finite set A and a relation R defined on the set, determine whether or not R is an equivalence relation on A.
RULE 24

If R satisfies Rules 21, 22, and 23, then R is an equivalence relation on A; otherwise it is not.

EXAMPLES 24

A. Set A: \[ \frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{1}{9}, \frac{2}{8}, \frac{3}{12}, \frac{2}{3}, \frac{4}{6} \]
   Relation R: equality ( = )

Answer:

Because the relation of equality on A satisfies Rules 21, 22, and 23, it is an equivalence relation on A.

B. Set A: \{1, 2, 3, 4, 5, 6, 7, 8\}
   Relation R: less than ( < )

Answer:

Because the relation, less than, does not satisfy Rule 22, it is not an equivalence relation on A.

EXERCISES 24

A. Set A: \{ \frac{6}{9}, \frac{9}{6}, \frac{20}{25}, \frac{3}{2}, \frac{10}{15}, \frac{4}{5}, \frac{2}{3} \}
   Relation R: equality ( = )

B. Set A: \{□, □, □, □, □\}
   Relation R: has same number of sides as

C. Set A: \{□, △, ▽, □, □\}
   Relation R: has the same number of sides as

D. Set A: \{1, 2, 3, 100/50, 5, 25/5, 7, 14/2, 9, 21/3\}
   Relation R: not equal to ( ≠ )

TASK 25

Given a set and an equivalence relation defined on the set, determine whether or not a pupil has attained the ideas of equivalence relation and equivalence class.

RULE 25

If the pupil is able to sort the set into equivalence classes by means of the relation, then he has attained the ideas of equivalence relation and equivalence class. Otherwise, he has not.
EXAMPLES 25

A. Given: \{101, 2, 25, 64, 302, 26, 54\} and the relation "has the same number of digits as"

Answer:

If the child forms the sets: \{2\}, \{25, 64, 26, 54\}, \{101, 302\} then he has attained the ideas of equivalence relation and equivalence class. Otherwise not.

B. Given: \{Δ, ⊤, ▽, □, △, ◐, ◙\} and the relation "has the same shape as"

Answer:

If the child forms the two sets \{Δ, ▽, ◐, ◙\}, \{⊤, □, △\}, then he has attained the ideas of equivalence relation and equivalence class. Otherwise not.

EXERCISES 25

A. \{2, 3, 4, 8, 9, 16, 27, 32\}, "is a power of the same number (2 or 3) as"
B. \{hit, fall, tall, a, I, king, sit, fit\}, "has the same number of letters as."
C. \{Jane, Judy, Jack, John, Jennifer\}, "is the same sex as".
D. \{doll, hit, small, bit, dog, log, fog, sit, fall\}, "has the same last two letters as."

TASK 26

Given a partition (i.e., set of equivalence classes) of some universe, find an equivalence relation which defines the partition.

RULE 26

Identify a common property of the elements in each equivalence class (by detecting a regularity in each). Then detect and describe a regularity among these properties. This regularity is the equivalence relation.

EXAMPLES 26

A. Given: \{Δ, ▽, ◐\}, \{□, □, ◐\}, \{◁, ◙\}

Answer:

The common properties are: "has three sides (is a triangle)," "has four sides," and "has five sides," respectively. The regularity (equivalence relation) is "has the same number of sides as."
Chapter 2
Section 5

B. Given: \{2, 8, 16, 32\}, \{3, 9, 27, 81\}

Answer: The common properties are: "is a power of 2" and "is a power of 3," respectively. The regularity (equivalence relation) is "is a power of the same number as."

EXERCISES 26

A. \{1, 3, 5, 7, 9\}, \{2, 4, 6, 8, 10\}
B. \{\text{\textbullet}, \text{\textcircled{1}}, \text{\textcircled{2}}\}, \{\text{\textcircled{1}}, \text{\textcircled{2}}, \text{\textcircled{3}}\}
C. \{x\mid x \text{ is a Male}\}, \{y\mid y \text{ is a Female}\}
D. \{1/2, 2/4, 3/6, 4/8\}, \{3/4, 6/8, 9/12, 12/16\}, \{1/1, 2/2, 3/3, 4/4\}
SECTION 6. Order Relations

TASK 27

Given a finite set A and a relation R, defined on the set, determine whether or not R is antisymmetric.

RULE 27

If for every pair of distinct elements x and y of set A, if x R y then y \( \not{\sim} \) x (i.e., y is not in relation R to x), then R is antisymmetric on A; otherwise it is not.

EXAMPLES 27

A. Set A: \{1, 2, 3, 4, 5\}
   Relation R: less than ( < )
   Answer:
   Because 1 < 2 but not 2 < 1, and this is true for each pair of numbers in the set, the relation "less than" is antisymmetric on A.

B. Set A: \{1, 2, 3\}
   Relation R: \{(1, 2), (1, 3), (2, 3), (3, 2)\}
   Answer:
   Because (2, 3) and (3, 2) are in R, R is not antisymmetric. (Because (1, 2) is in R, but (2,1) is not, the relation is not symmetric either.)

EXERCISES 27

A. Set A: \{12/3, 4/2, 4, 8/2\}
   Relation R: inequality ( \( \not{\sim} \) )
B. Set A: \{[a, b, c], [a, b], [b]\}
   Relation R: is a subset of (\( \subseteq \))
C. Set A: \{4, 5, 6, 7\}
   Relation R: \{(4, 5), (5, 6), (6, 7)\}
D. Set A: \{6, 7, 8, 9\}
   Relation R: \{(9, 8), (8, 7), (7, 6), (7, 8)\}

TASK 28

Given a finite set A and a relation R defined on the set, determine whether or not R is irreflexive.
RULE 28

For every element x of set A, if \( x \not\sim x \) (i.e., if x is not related to itself), then R is irreflexive on A; otherwise it is not.

EXAMPLES 28

A. Set A: \( \{1, 2, 3\} \)
   Relation R: less than ( < )

Answer:

Because 1 < 1, 2 < 2, and 3 < 3 are all false, "less than" is irreflexive on A.

B. Set A: The brothers Tom, Dick, and Harry
   Relation R: is a brother of

Answer:

Because no one is a brother of himself, "is a brother of" is irreflexive.

C. Set A: \( \{1, 2, 3\} \)
   Relation R: \( \{(1, 1), (2, 2), (2, 3)\} \)

Answer:

Because (2, 2) is in R, R is not irreflexive on A. (Because (3, 3) is not in R, R is not reflexive either.)

EXERCISES 28

A. Set A: A row of children in a classroom (from front to back: Tom, Sally, Sandra, Richard, James).
   Relation R: sits behind

B. Set A: \( \{\{A\}, \{B\}, \{A, B\}\} \)
   Relation R: is a subset of (C)

C. Set A: \( \{3, 4, 5, 6\} \)
   Relation R: \( \{(3, 3), (4, 4), (5, 5), (6, 6)\} \)

D. Set A: \( \{3, 4, 5, 6\} \)
   Relation R: \( \{(3, 3), (3, 4), (4, 3), (5, 5), (5, 6), (6, 5), (6, 6)\} \)

TASK 29

Given a finite set A and a relation R defined on the set, determine whether or not R is an order relation.
RULE 29

If \( R \) is antisymmetric (Rule 27) and transitive (Rule 23), then \( R \) is an order relation; otherwise it is not. Note: The ordering imposed on a set by an order relation can be represented by a lattice (i.e., by a (possibly) branching network of points and lines between them.)

EXAMPLES 29

A. Set \( A: \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \)
   Relation \( R: \) less than (\(<\))

   Answer:
   Because Rule 27 is satisfied for each pair of numbers and Rule 23 is satisfied for each triple of numbers, "less than" is an order relation on the set. It can be represented: \( 1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 9 < 10 \).

B. Given: \( \{1, 2, 3, 4, 5\} \)
   Relation \( R: \) unequal to (\(\neq\))

   Answer:
   Because \( 2 \neq 4 \) and \( 4 \neq 2 \) but not \( 2 \neq 2 \) (i.e., \( 2 = 2 \)), Rule 23 is not satisfied and \( \neq \) is not an order relation.

EXERCISES 29

A. Set \( A: \{\triangle, \square, \bigcirc, \bigotimes\} \)
   Relation \( R: \) has less sides than

B. Set \( A: \{1, 2, 3, 4, 5\} \)
   Relation \( R: \) greater than or equal to (\(\geq\))

C. Set \( A: \{\{a, b, c\}, \{a, b\}, \{a, b\}, \{a\}\} \)
   Relation \( R: \) is a subset of

D. Set \( A: \{x | x \) is a state of the United States bordering the Pacific Ocean\} \)
   Relation \( R: \) has less population than.

---

TASK 30

Given a finite set \( A \) and an order relation \( R \) (Rule 29), determine whether \( R \) is a linear or partial order relation on the set, and represent the relation by a graph.

RULE 30

If for every pair of distinct elements \( x \) and \( y \) in the set, either \( x R y \) or \( y R x \), then \( R \) is a linear order relation, otherwise it is a partial order relation. The ordering imposed on a set by a linear order relation can be represented on a line,
that imposed by a partial order relation by a branching graph.

EXAMPLES 30

A. Set A: \{1, 2, 3, 4, 5\}
Relation R: less than (\(<\))

Answer:

"less than" is a linear order relation because for each pair of numbers \(x, y\) either \(x < y\) or \(y < x\). The relation can be represented: \(1 < 2 < 3 < 4 < 5\)

B. Set A: \{\{a, b, c\}, \{a, b\}, \{a, c\}, \{a\}, \{b\}, \{c\}\}
Relation R: is a subset of

Answer:

The order relation, \(\subseteq\), is partial for this set because for \{a\} and \{b\}, neither \{a\} \(\subseteq\) \{b\} nor \{b\} \(\subseteq\) \{a\}. The relation can be represented:

```
<table>
<thead>
<tr>
<th>a, b, c</th>
</tr>
</thead>
<tbody>
<tr>
<td>a, b</td>
</tr>
<tr>
<td>a, c</td>
</tr>
<tr>
<td>b</td>
</tr>
<tr>
<td>a</td>
</tr>
<tr>
<td>c</td>
</tr>
</tbody>
</table>
```

where "\(\rightarrow\)" is "\(\subseteq\)".

EXERCISES 30

A. Set A: \{\{a, b, c, d\}, \{a, b, c\}, \{a, b\}, \{a\}\}
Relation R: is a subset of

B. Set A: \{\{x, y, z\}, \{x, z\}, \{y, z\}, \{x\}, \{y\}, \{z\}\}
Relation R: is a subset of

C. Set A: \{x| x is a state of the continental United States bordering the Pacific Ocean\}
Relation R: has less population than

D. Set A: \{dog, king, am, fight, a, report\}
Relation R: has less letters than


TASK 31

Given a finite set \(A\) and a linear or partial order relation \(R\) (Rule 30), determine whether \(R\) is strict or non-strict on the set, and represent the relation by a graph.
RULE 31

If R is irreflexive (Rule 28), then it is a strict linear or partial order relation on the set; if R is not irreflexive then it is a non-strict linear or partial order relation. The graph representation is the same as in Rule 30 where elements can be repeated in the non-strict order relation.

EXAMPLES 31

A. Set A: \{child, mother, father, maternal grandmother, maternal grandfather, paternal grandmother, paternal grandfather\}
   Relation R: is an ancestor of
   Answer:
   "Is an ancestor of" is a strict (by Rule 31), partial (by Rule 30), order relation on the set. The relation can be expressed by a branching graph corresponding to the "family tree."

B. Set A: \{[a, b], [a], [b]\}
   Relation R: is a subset of \((\subseteq)\)
   Answer:
   "Is a subset of" is a non-strict (by Rule 31 because \([a] \subseteq [a]\) is true), partial (by Rule 30), order relation. The relation can be expressed by the branching graph:

C. Set A: \{1, 2, 3\}
   Relation R: less than \((<)\)
   Answer:
   "Less than" is a strict (by Rule 31), linear (by Rule 30), order relation on
the set. This can be expressed by the line graph:

\[ 1 \leq 2 \leq 3 \]

D. Set A: \([\{a, b, c\}, \{a, b\}, \{a\}]\)
Relation R: is a subset of (\(c\))

Answer:

"is a subset of" is a non strict (by Rule 31) linear (by Rule 30) order relation on this set. It can be expressed by the line graph:

\[ \{a\} \subset \{a, b\} \subset \{a, b, c\} \]

EXERCISES 31

A. Set A: \([1, 2, 3, 4, 5]\)
Relation R: greater than or equal to (\(\geq\))
B. Set A: \([\{a, b, c, d\}, \{a, b, d\}, \{a, c, d\}, \{a, b\}, \{a, c\}, \{a\}, \{c\}]\)
Relation R: is a subset of (\(c\))
C. Set A: \([9, 71, 3, 57, 192]\)
Relation R: less than (\(<\))
D. Set A: \([a, b, d, e, c, g, z, w, v]\)
Relation R: is in alphabetical order to
E. Set A: \([\{a, b, c, d\}, \{b, c, d\}, \{b, d\}, \{b\}\]}
Relation R: is a subset of (\(c\))
F. Set A: \([\text{highest, fight, high, my, low, i}\])
Relation R: has the same or less number of letters than
G. Set A: \([\text{Ant, An, A, Nats,At}\])
Relation R: can be spelled using only letters from

---

TASK 32

Given a finite set A and an order relation R, determine whether or not a pupil fully understands the order relation on the given set.

RULE 32

If the child can:

a) give the successors, if any, from the set to any given element in the set according to the order relation,
b) represent the set, using the order relation, in a linear or branching graph, and
c) name the type of order relation imposed on the set, then he fully understands the order relation on the given set.
EXAMPLES 32

A. Set A: \{1, 2, 3, 4, 5, 6\}
   Relation R: less than (\(<\))

Answer:

If the child
   a) gives 2 as the successor of 1, 3 as the successor of 2, etc,
   b) writes: 1 < 2 < 3 < 4 < 5 < 6,
   c) states that "less than" is a strict linear order relation on
this set,

then he understands the order relation "less than" with respect to set A.

B. Set A: \{\{a, b\}, \{a, b, c\}, \{a, c\}, \{a\}, \{b\}, \{c\}\}
   Relation R: is a subset of (\(\subseteq\))

Answer:

If the child
   a) gives \{a, c\} and \{a, b\} as the successors of \{a\},
   or \{a, b, c\} as the successor of \{a, b\}, etc,
   b) writes: (the possible graph)

\begin{center}
\begin{tikzpicture}
  \node (a) at (0,0) {\{a, b, c\}};
  \node (b) at (0,-1) {\{a, b\}};
  \node (c) at (0,-2) {\{a, c\}};
  \node (d) at (1,-1) {\{b\}};
  \node (e) at (-1,-1) {\{a\}};
  \node (f) at (-1,-2) {\{c\}};
  \draw[->] (a) -- (b);
  \draw[->] (a) -- (c);
  \draw[->] (b) -- (d);
  \draw[->] (b) -- (e);
  \draw[->] (c) -- (f);
\end{tikzpicture}
\end{center}

where "\(\rightarrow\)" = "\(\subseteq\)".

   c) states that "is a subset of" is a non strict partial order
relation on this set,

then he understands the order relation "is a subset of" with respect to set A.

EXERCISES 32

A. Set A: \{\{a, b, c, d, e\}, \{a, b, c, e\}, \{a, c, e\}, \{e\}\}
   Relation R: is a subset of

B. Set A: \{son, mother, son's daughter, son's son, son's son's son, daughter,
   daughter's son, daughter's daughter\}
   Relation R: is a descendant of

C. Set A: \{a, b, e, g, f, h, d, i, x, v\}
   Relation R: is in alphabetical order to

D. Set A: \{-3, 6, 9, 0, -6, 13, 10\}
   Relation R: is less than or equal to (\(\le\))
SECTION 7. Binary Operations

TASK 33

Given an operation \( \oplus \), determine whether or not it is binary and, if it is binary, give a representation of it.

RULE 33

If two elements \( x \) and \( y \) are combined by operation \( \oplus \) to give a unique third element \( z \), then \( \oplus \) is binary and can be represented \( (x, y) \underset{\oplus}{\rightarrow} z \). Otherwise it is not binary.

EXAMPLES 33

A. Given: "addition of pairs of numbers."
Answer:

Because addition combines two numbers to produce a unique third number, it is binary and can be represented: \( (n, m) \underset{\oplus}{\rightarrow} n + m \).

B. Given: "finding the average of three numbers"
Answer:

Because three elements are needed to perform this operation, it is not binary.

EXERCISES 33

A. "Find the average of two numbers"
B. "Multiply pairs of numbers"
C. "Given length and width find the area"
D. "Given length, width, and height, find the volume"

TASK 34

Given a binary operation \( \oplus \), represent \( \oplus \) as a ternary relation.

RULE 34

Given \( (x, y) \underset{\oplus}{\rightarrow} z \), write \( (x, y, z) \).

EXAMPLES 34

A. Given: \( (n, m) \underset{\oplus}{\rightarrow} n + m \) Write: \( (n, m, n + m) \)
B. Given: \( (n, m) \underset{\oplus}{\rightarrow} n - m \) Write: \( (n, m, n - m) \)
EXERCISES 34

A. \((n, m) \rightarrow n \cdot m\)
B. \((r_1, r_2) \rightarrow r_1 \text{ followed by } r_2\)
C. \((mn, n) \rightarrow m\)

---

TASK 35

Given a finite set \(A\) and a binary operation on the set, determine whether or not a pupil has mastered the given operation.

RULE 35

Present the pupil with a wide variety of pairs of elements from the set. If he can in each case determine the unique third element specified by the binary operation, then he has mastered the binary operation on the set, otherwise he has not.

EXAMPLES 35

A. Set \(A\): \([1, 2, 3, 4, 5]\)
   Binary operation: add pairs of elements

   Answer:
   
   Present to the pupil \((1, 2) \rightarrow , (2, 2) \rightarrow , , (3, 5) \rightarrow , etc. \) If the pupil answers \(3, 4, 8, \) etc. then he has mastered the operation on the set.

B. Set \(A\): \([9, 5, 7, 13]\)
   Binary operation: average pairs of numbers

   Answer:
   
   Present to the pupil \((9, 5) \rightarrow , (9, 13) \rightarrow , (13, 9) \rightarrow . \) If the pupil answers \(7, 11, 36, \) then he has mastered the operation on the set.

EXERCISES 35

A. Set \(A\): \([7, 9, 11, 13]\)
   Binary operation: multiply pairs of elements
B. Set \(A\): \([7, 9, 11, 13]\)
   Binary operation: average pairs of elements
C. Set \(A\): \([\text{length 6"}, \text{width 8"}]\)
   Binary operation: find the area
D. Set \(A\): \([3, 4, 5]\)
   Binary operation: \(\oplus\) where \(x \oplus y = x \cdot y\)
SECTION 1. Logical Possibilities

TASK 1

State the set of all logical possibilities for a given situation.

RULE 1

List in set notation all elements (entities) which could possibly satisfy the given situation.

EXAMPLES 1

A. The set of logical possibilities for the state of the weather, might be
   \{no precipitation, precipitation\}.  
B. The set of logical possibilities for names of the months is 
   \{January, February, March, April, May, June, July, August, September, October, 
   November, December\}.  
C. The set of logical possibilities for the number of days in a month is 
   \{28, 29, 30, 31\}.

EXERCISES 1

A. the number of days in the year 
B. colors in a stop-light 
C. major directions on a compass

---

TASK 2

Given a string of words, determine whether or not it is a statement.
RULE 2

If the words form a sentence which can be classified as either true or false, then it is a statement. Otherwise it is not. (Statements are also called "assertions").

EXAMPLES 2

The following are statements:

A. Louis IX was king of England. (false)
B. There are seven days in a week. (true)
C. Water is composed of nitrogen and oxygen. (false)

The following are not statements:

D. John and Mary are
E. x is a natural number (depends on what x is).
F. What is the height of the Empire State Building?

EXERCISES 2

A. The time is 8 o'clock.
B. When do we leave?
C. _______ is a member of the National Guard.
D. Every man is mortal.
E. Good morning!
F. The sum of 60 and 27 is 58.

EXERCISES 2

A. The time is 8 o'clock.
B. When do we leave?
C. _______ is a member of the National Guard.
D. Every man is mortal.
E. Good morning!
F. The sum of 60 and 27 is 58.

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Chapter 3
Section 1

D. The name of the month starts with J and has 6 letters. (false).

The following are neither true nor false.

E. It is raining. (depends on the weather)
F. There are 6 letters in the name for today (day of the week). (depends on what day of the week it is)
G. In San Francisco there was at least one inch of rain last month. (depends on what the rainfall in San Francisco actually was last month)

EXERCISES 3

A. $x = 6$ or $x \neq 6$.
B. The dog has spots and it does not have spots.
C. The name of the month has more than two letters.
D. The population of the state is less than three million.
E. The cat caught the mouse or the cat didn't catch the mouse.
SECTION 2. Observation Versus Deduction

TASK 4

Given that the statements "If P, then Q" and "P," are true, state the conclusion which follows.

RULE 4

If "If P, then Q" and "P" are true, the logical rule modus ponens says that "Q" is true. "If P, then Q" and "P" are called premises, and "Q" is called the conclusion.

EXAMPLES 4

A. Premises:
   1. If John goes to the store, then he will buy milk.
   2. John goes to the store.

   Conclusion:
   He will buy milk.

B. Premises:
   1. The cow gave milk.
   2. If the cow gave milk, then mother made cream.

   Conclusion:
   Mother made cream.

C. Premises:
   1. If the truck got a flat tire, then it was driven off the road.
   2. The truck got a flat tire.

   Conclusion:
   The truck was driven off the road.

EXERCISES 4

A. Premises:
   1. If the soup is hot, then we will eat.
   2. The soup is hot.
B. Premises:
1. It is three o'clock.
2. If it is three o'clock, then the train has arrived.

C. Premises:
1. When John arrives, we will leave for Grandmother's.
2. John has arrived.
SECTION 3. Fallacies

TASK 5

Given an argument which appears to use the logical form modus ponens, identify and construct the missing or hidden premises.

RULE 5

Compare the premise statements in the given argument to the form modus ponens and identify which premise is missing. Then state the missing premise.

EXAMPLES 5

A. Given: John Doe received the most votes in the senatorial election. Therefore what he says must be true.

Answer:

If P then Q
P
Therefore Q

If John Doe received the most votes in the senatorial election, therefore what he says must be true.

Construct the missing premise:

If John Doe received the most votes in the senatorial election then what he says must be true.

B. Given: Detergent A makes more suds than detergent B; therefore detergent A is better than detergent B.

Answer:

If P then Q
P
Therefore Q

If detergent A makes more suds than detergent B, therefore detergent A is better than detergent B.

Construct the missing premise:

If detergent A makes more suds than detergent B, then detergent A is better than detergent B.

Note: In each example, the missing premise is not necessarily true. Hence, the conclusion is not necessarily true.

EXERCISES 5

A. Author A publishes more than author B; therefore author A is a better writer
than author B.

B. Fewer of surgeon A's patients die than surgeon B's; therefore, surgeon A is more skilled than surgeon B.

C. Mr. A's house is not as well maintained as Mr. B's; therefore, Mr. A is not as good a person as Mr. B.

D. Gasoline A has additive X which gasoline B does not have; therefore, gasoline A is better than gasoline B.

---

TASK 6

Given an argument, determine whether or not it involves reasoning from the converse.

RULE 6

Compare the premises in the given argument with the form, "If P then Q" and "Q". Then, if the conclusion is "P", the argument involves invalid reasoning from the converse; otherwise it does not.

EXAMPLES 6

A. Given: A number divisible by 4 is even. I am thinking of an even number; therefore I am thinking of a number divisible by 4.

Answer:

<table>
<thead>
<tr>
<th>Modus Ponens</th>
<th>Argument</th>
<th>Form of Argument</th>
</tr>
</thead>
<tbody>
<tr>
<td>If P then Q</td>
<td>If a number is divisible by 4 then it is even.</td>
<td>If P then Q</td>
</tr>
<tr>
<td>P</td>
<td>I am thinking of an even number,</td>
<td>Q</td>
</tr>
<tr>
<td>Therefore Q</td>
<td>Therefore, I am thinking of a number divisible by 4.</td>
<td>Therefore P</td>
</tr>
</tbody>
</table>

Since the argument has premises "If P then Q" and "Q" rather than "If P then Q" and "P" it involves invalid reasoning from the converse.

B. Given: All the intelligent people in town are supporting the bond issue. Mr. Jones is supporting the bond issue; therefore Mr. Jones is an intelligent person.

Answer:

<table>
<thead>
<tr>
<th>Modus Ponens</th>
<th>Argument</th>
<th>Form of Argument</th>
</tr>
</thead>
<tbody>
<tr>
<td>If P then Q</td>
<td>If a man is intelligent then he will support the bond issue.</td>
<td>If P then Q</td>
</tr>
<tr>
<td>P</td>
<td>Mr. Jones supports the bond issue,</td>
<td>Q</td>
</tr>
<tr>
<td>Therefore Q</td>
<td>Mr. Jones is intelligent,</td>
<td>Therefore P</td>
</tr>
</tbody>
</table>
Since the argument has premises "If P then Q" and "Q" rather than "If P then Q" and "P" it involves invalid reasoning from the converse.

EXERCISES 6

A. "Real men smoke Muscle Cigarettes. I smoke Muscle Cigarettes; therefore I am a real man."

B. "All beautiful women use Seduction No. X Face Cream. I use Seduction No. X Face Cream; therefore, I am beautiful."

C. All communists support the United Nations. John supports the United Nations; therefore, John is a communist.
SECTION 4. Axiomatizing

TASK 7

Given a list of assertions, shorten it so that the assertions deleted can be deduced from those remaining.

RULE 7

Examine each assertion in the list. If an assertion can be deduced from the remaining assertions (e.g., as in Rule 4), delete it. Continue until no remaining statements can be deleted.

EXAMPLES 7

A. Given:

1. All pro football players are good athletes.
2. My cousin is a pro football player.
3. My cousin is a good athlete.

Answer:

Delete statement 3 because it can be deduced from statements 1 and 2.

B. Given:

1. All good students get 3.5 averages or better.
2. I got a 3.5 average or better.
3. I am a good student.

Answer:

Delete statement 2 because it can be deduced from statements 1 and 3.

C. Given:

1. Sam is older than Tom.
2. Tom and Elizabeth are twins.
3. Sam is older than Elizabeth.
4. Tom and Elizabeth are the same age.

Answer:

Statements 3 and 4 may be deleted because statement 3 can be deduced from statements 1 and 2, and statement 4 follows from statement 2.

EXERCISES 7

A. Tommy and I are thinking of the same number.
   I am thinking of the number 20.
Tommy is thinking of the number 20.

B. Exactly one third of the people in my class are boys.
   There is at least one boy in my class.
   There are more girls in my class than boys.
   There are at least two girls in my class.

C. Sam is older than Tom.
   Tom and Elizabeth are twins.
   Sam is older than Elizabeth.
   Tom and Elizabeth are the same age.
   Sam is younger than Susan.
   Susan is older than Tom.
   Elizabeth is younger than Susan.
SECTION 5. Deductive Patterns

TASK 8

Given statements of the form "Either P or Q" and "not P" ("not Q"), determine the logical conclusion.

RULE 8

If "Either P or Q" and "not P" are true statements, called premises, the logical conclusion is "Q."
If "Either P or Q" and "not Q" are the premises, then the conclusion is "P."

EXAMPLES 8

A. Premises:
   1. Either John will return the books to the library by the date they are due or he will pay the fine.
   2. He will not return them by the date due.

   Conclusion:
   John will pay the fine.

B. Premises:
   1. I will not go to the football game.
   2. Either I will go to the picnic or I will go to the football game.

   Conclusion:
   I will go to the picnic.

C. Premises:
   1. Either John is sick or he passed the physical examination.
   2. John is not sick.

   Conclusion:
   John passed the physical examination.

EXERCISES 8

A. 1. Either Sue is home by 4:30 PM or she will be disciplined.
   2. Sue is not home by 4:30 PM.
B. 1. Either I go to school tomorrow or tomorrow is a holiday.
   2. Tomorrow is not a holiday.
C. 1. Either Odette went to the movie or she had a date.
   2. Odette did not have a date.
D. 1. I did not get at least a "70" on the test.
    2. Either I get at least "70" on the test or I fail the course.

TASK 9

Given statements of the form "All P's are Q's" (where P and Q are sets having specific properties) and "x is a P", determine the logical conclusion.

RULE 9

For the premises "All P's are Q's" and "x is a P," "x is a Q" is the logical conclusion.

EXAMPLES 9

A. Premises:
   1. All dogs will someday die.
   2. Fido is a dog.

   Conclusion:
   Fido will someday die.

B. Premises:
   1. This window is a square.
   2. Every square is a rectangle.

   Conclusion:
   This window is a rectangle.

C. Premises:
   1. Every player in this tournament is an amateur.
   2. Jack is playing in this tournament.

   Conclusion:
   Jack is an amateur.

EXERCISES 9

A. 1. All Supreme Court Justices are appointed for life.
    2. Mr. X is a Supreme Court Justice.

B. 1. All pentagons have five sides.
    2. This figure is a pentagon.
C. 1. Jack is a member of the Blue Party.
   2. All members of the Blue Party are voting for reform.

D. 1. Every team member's injury will hurt the team.
   2. Jackson is on the team and is injured.

Task 10

Given an argument which appears to be in the logical form of Rules 8 and 9, determine whether or not it uses these forms incorrectly.

Rule 10

A. (For Rule 8) Compare the premise statements in a given argument to the form "Either P or Q" and "not P" ("not Q") if the argument does not follow this form then it involves invalid reasoning; otherwise it does not.

B. (For Rule 9) Same as Rule 10A except substitute "All P's are Q's" and "x is a P" for "Either P or Q" and "not P" (or "not Q").

Examples 10

A1. Given:

   Either John is sick or he will pass his physical examination. John is sick. Therefore he will not pass his physical examination.

   Answer:

   \[
   \begin{array}{ccc}
   \text{Correct Form} & \text{Argument} & \text{Form of Argument} \\
   \text{Either P or Q} & \text{Either John is sick or he will pass his physical examination.} & \text{Either P or Q} \\
   \text{Not P} & \text{John is sick.} & \text{P} \\
   \text{Therefore Q} & \text{Therefore he will not pass his physical examination.} & \text{Therefore not Q}
   \end{array}
   \]

   Since the argument has form "Either P or Q" and "P" rather than "Either P or Q" and "not P," it involves invalid reasoning.

A2. Given:

   Either Joan has a date or she went to the movies. She went to the movies. Therefore she did not have a date.

   Answer:

   \[
   \begin{array}{ccc}
   \text{Correct Form} & \text{Argument} & \text{Form of Argument} \\
   \text{Either P or Q} & \text{Either Joan has a date or she went to the movies.} & \text{Either P or Q}
   \end{array}
   \]

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not Q  
Therefore P  
Q  
Therefore she did not have a date.

Since the argument has form "Either P or Q" and "Q" rather than "Either P or Q" and "not Q," it involves invalid reasoning.

Bl. Given:

All rectangles have opposite sides parallel. This figure has opposite sides parallel. Therefore it is a rectangle.

Answer:

<table>
<thead>
<tr>
<th>Correct Form</th>
<th>Argument</th>
<th>Form of Argument</th>
</tr>
</thead>
<tbody>
<tr>
<td>All P's are Q's</td>
<td>All rectangles have opposite sides parallel.</td>
<td>All P's are Q's</td>
</tr>
<tr>
<td>x is a P</td>
<td>This figure has opposite sides parallel.</td>
<td>x is a Q</td>
</tr>
<tr>
<td>Therefore x is a Q</td>
<td>Therefore it is a rectangle,</td>
<td>Therefore it is a P</td>
</tr>
</tbody>
</table>

Since the argument has form "All P's are Q's" and "x is a Q" rather than "All P's are Q's" and "x is a P," it involves invalid reasoning.

EXERCISES 10

A. Either George was not at home or he was not at school. George was not at school. Therefore, George was at home.
B. All third grade girls passed the test. Sue passed the test. Therefore, Sue is a third grade girl.
C. Either John was sick or he forgot what time it was. John forgot the time. Therefore, John was not sick.
D. Either this is the wrong room or I came on the wrong day. I did not come on the wrong day. Therefore, this is the wrong room.
E. All Pontiacs are made by General Motors Company. Wally's new car is a Pontiac. Therefore, Wally's car is made by General Motors.
F. All rectangles have four sides. This figure has four sides. Therefore, this figure is a rectangle.
G. All the boys in the fourth grade played on the softball team. John played on the softball team. Therefore, John is in the fourth grade.

\[ \text{TASK 11} \]

Given an argument which appears to use the logical form (A) "Either P or Q" and "Not P" (or "not Q") (Rule 8) or (B) "All P's are Q's" and "x is a P" (Rule 9), identify which premise is missing. Then state the missing premise.

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RULE 11

A. Apply Rule 5, substituting "Either P or Q" and "not P" (or "not Q") for the logical form of modus ponens.
B. Apply Rule 5, substituting "All P's are Q's" and "x is a P" for the logical form of modus ponens.

EXERCISES 11

A. James is not late; so he will not be disciplined.
B. Debby did not have her money in on time; so she cannot go on the trip.
C. The clock is wrong; therefore I am late.
D. It is raining; therefore the sprinkler is not on.
E. John is a man; therefore John will die.
F. Mr. N. is a politician; so Mr. N. is dishonest.
G. Jane was the first applicant; so she will get the prize.
SECTION 6. Set Operations and Venn Diagrams

TASK 12

Given two finite sets A and B such that A is a subset of B (A ⊆ B) (Rule 4, Chapter 2), determine whether or not A is a proper subset of B.

RULE 12

Check the elements of both sets. If there is at least one element in B which is not in A, then A is a proper subset of B; otherwise it is not.

EXAMPLES 12

A. Given: {1, 2, 3} ⊆ {1, 2, 3, 4}.

Answer:

The first set is a proper subset of the second since there is an element (4) in the second set that is not also in the first set.

B. Given: {□, ○, *} ⊆ {□, ○, *}.

Answer:

The first set is not a proper subset of the second since there is no element in the second set not also in the first.

EXERCISES 12

A. {x, y, z} ⊆ {w, x, y, z}
B. {x; x is a State touching the Pacific Ocean} ⊆ {Washington, Oregon, California, Alaska, Hawaii}
C. {1, 2, 3, 4} ⊆ {1, 2, 3, 4, 7}

TASK 13

Use a Venn diagram to represent (three) ways in which two subsets, D and E, of a universal set, U, may be related.

RULE 13

A. Sets D and E may have no elements in common. In this case D and E are called "disjoint" sets and are represented as in A below.

B. All elements of one set, D, may be elements of the other set E. In this case, D is a subset of E and they are represented as in B below.

C. If sets D and E satisfy neither A nor B above, then they are represented as in C below.
EXAMPLES 13

A. D and E are disjoint

B. D is a subset of E

C. D and E have some (but not all) elements in common

EXERCISES 13

Identify the relation between sets D and E in each of the following.

A.

B.

C.

D.

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TASK 14

Given two finite sets, $A$ and $B$, form the union $A \cup B$, and use a Venn diagram to represent the union.

RULE 14

If an element $x$ is in set $A$ or in set $B$ or both, then $x$ is in the union, $A \cup B$. Represent sets $A$ and $B$ in the same Venn diagram (Rule 13). Shade all parts of the diagram which are inside the figure for $A$, or the figure for $B$, or both. This shaded area represents $A \cup B$.

EXAMPLES 14

A. Given: $A = \{a, b, c\}, B = \{c, d, e\}.$

Answer: $A \cup B = \{a, b, c, d, e\}$

Note: Even though $c$ is listed in set $A$ and in set $B$ it is listed only once in set $A \cup B$.

B. Given: $A = \{1, 2, 5, 6\}, B = \{3, 4, 7, 9\}$

Answer: 

$A \cup B = \{1, 2, 3, 4, 5, 6, 7, 9\}$

C. Given: $A = \{1, 2, 3\}, B = \{2, 3\}$

Answer: 

$A \cup B = \{1, 2, 3\}$
EXERCISES 14

A. \( A = \{ \bigcirc, \Box, \# , * \}, B = \{ * , \bigcirc, \triangle \} \)
B. \( A = \{10, 12, 8, 13\}, B = \{9, 11, 14\} \)
C. \( A = \{ \bigcirc, \triangle, \Box \}, B = \{ \bigcirc, \boxtimes, \# , \triangle \} \)

TASK 15

Given two finite sets, \( A \) and \( B \), form the intersection, \( A \cap B \), and use a Venn diagram to represent the intersection.

RULE 15

If an element \( x \) is in both sets \( A \) and \( B \), then \( x \) is in the intersection, \( A \cap B \). Represent sets \( A \) and \( B \) in the same Venn diagram (Rule 13). Shade the area common to both sets \( A \) and \( B \). This shaded area represents \( A \cap B \).

EXAMPLES 15

A. Given: \( A = \{ \triangle, \bigcirc, \Box \}, B = \{ \triangle, \Box, \# \} \)
Answer:
\[ A \cap B = \{ \triangle, \Box \} \]
B. Given: \( A = \{ \triangle, \bigcirc, \# \} \) \( B = \{ \triangle, \# \} \)

Answer: \( A \cap B = \{ \triangle, \# \} \)

\[
\begin{array}{c}
\text{U} \\
A \cap B \text{ (is shaded)}
\end{array}
\]

C. Given: \( A = \{1, 2, 3\} \) \( B = \{7, 8, 9\} \)

Answer: \( A \cap B = \{ \} \)

Note: \( \{ \} \) means "the empty set" or set with no elements.

\[
\begin{array}{c}
\text{U} \\
A \cap B = \emptyset \text{ (no shading)}
\end{array}
\]

EXERCISES 15

A. \( A = \{1, 2, 3\} \) \( B = \{2, 3, 4\} \)
B. \( A = \{3, 5, 2, 1\} \) \( B = \{10, 20, 1, 5\} \)
C. \( A = \{a, b, c\} \) \( B = \{b, a, c, d, g\} \)
D. \( A = \{ \triangle, \bigcirc, \square \} \) \( B = \{ \square, *, \# \} \)
E. \( A = \{1, 3, 5, 7\} \) \( B = \{9, 6\} \)
F. \( A = \{12, 13, 15, 16\} \) \( B = \{13\} \)

TASK 16

Given two finite sets \( A \) and \( B \), determine the difference sets \( B - A \) and \( A - B \) and use a Venn diagram to represent the difference set.

RULE 16

For each element in \( B \) determine if the element is in \( A \). If the element is in \( A \), exclude it from the difference set, \( B - A \). Otherwise, include it in the difference set. Interchange \( A \) and \( B \) in the above rule to obtain the difference set, \( A - B \).
Represent sets $A$ and $B$ in the same Venn diagram (Rule 13). For $B - A$ shade the area inside of $B$ which is not in $A$ (for $A - B$ shade area inside $A$ and not in $B$). It is worth noting that $B - A$ is $B \cap \overset{\sim}{A}$ (and $A - B$ is $A \cap \overset{\sim}{B}$).

**EXAMPLES 16**

A. Given: $A = \{a, b, c\}$ $B = \{b, c, d, e\}$

Answer: $B - A = \{d, e\}$  
$A - B = \{a\}$

B. Given: $A = \{1, 2, 3\}$ $B = \{4, 5, 6\}$

Answer: $B - A = \{4, 5, 6\}$  
$A - B = \{1, 2, 3\}$

C. Given: $A = \{\triangle, \square, *, \#\}$ $B = \{\square, *\}$

Answer: $B - A = \{\} = \emptyset$  
$A - B = \{\triangle, \#\}$

$B - A = \emptyset$ (no shading)  
$A - B$ (is shaded)
EXERCISES 16

A. $A = \{a, b, c, d, e\}$  $B = \{c, e, f\}$
B. $A = \{1, 3, 9, 2, 6\}$  $B = \{2, 9, 1\}$
C. $A = \{\Delta, \Box, \square, \#\}$  $B = \{\bigcirc, \Diamond\}$
D. $A = \{\text{dog, cat, mouse}\}$  $B = \{\text{mouse, giraffe, cow}\}$
E. $A = \{20, 21, 24, 30\}$  $B = \{20, 30, 21, 29, 24\}$
F. $A = \{x \mid x \text{ is a United States Senator}\}$  $B = \{y \mid y \text{ is a United States Congressman}\}$. 

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SECTION 7. The Statement Logic

TASK 17

Given a statement $p$, form its negation and represent the negation symbolically.

RULE 17

To form the negation of a statement, put the phrase "It is not the case that" before the statement. If $p$ represents the statement, write $\neg p$ for the negation.

EXAMPLES 17

A. Given: $p$: "It is raining."

Answer:

$\neg p$: "It is not the case that it is raining." or equivalently, "It is not raining."

B. Given: $p$: "John is at home."

Answer:

$\neg p$: "It is not the case that John is at home." or equivalently, "John is not at home."

EXERCISES 17

A. "The door is closed."
B. "The dog can swim."
C. "The house shakes."
D. "Jane is not running."

TASK 18

Given two statements $p$ and $q$, form a compound statement using the connective and. Write a symbolic representation of the compound statement. If the truth or falsity of the two statements is known, determine the truth or falsity of the compound statement.

RULE 18

To form the compound statement, place "and" between the two statements. If the statements are represented by $p$ and $q$, represent the compound statement symbolically by writing $p \land q$. Finally, if both $p$ and $q$ are true statements, then statement $p \land q$ is a true statement. If either $p$ or $q$ or both $p$ and $q$ are false, then $p \land q$ is false.
EXAMPLES 18

A. Given: p: "It is raining."
   q: "The windows are closed."

Answer: p \( \land \) q: "It is raining and the windows are closed."

Suppose it is raining and the windows are closed. Then both statements are true, and so p \( \land \) q is true. Suppose it is not raining and the windows are closed; then only one statement is true, and p \( \land \) q is false.

B. Given: p: "John is home."
   q: "Bill is at school."

Answer: p \( \land \) q: "John is at home and Bill is at school."

Suppose John is home, but Bill is not at school. Then only one statement is true; so p \( \land \) q is false. If John also is not at home, then neither statement is true; so p \( \land \) q is false. Remember, p \( \land \) q is true only when both p and q are true.

EXERCISES 18

Write out p \( \land \) q, and determine if p \( \land \) q is true or false depending on the truth or falsity of p and q as indicated in the exercise.

A. p: "Snow is falling."
   q: "The wind is blowing."

   If p is true and q is true, then p \( \land \) q is _______ (true or false).
   If p is false and q is true, then p \( \land \) q is _______ (true or false).
   If p is true and q is false, then p \( \land \) q is _______ (true or false).
   If p is false and q is false, then p \( \land \) q is _______ (true or false).

B. p: "I am going to the store."
   q: "You are not coming along."

   If p is false and q is true, then p \( \land \) q is _______ (true or false).
   If p is false and q is false, then p \( \land \) q is _______ (true or false).
   If p is true and q is true, then p \( \land \) q is _______ (true or false).
   If p is true and q is false, then p \( \land \) q is _______ (true or false).

C. p: "The door is not locked."
   q: "The door is closed."

   If p is true and q is false, then p \( \land \) q is _______ (true or false).
   If p is true and q is true, then p \( \land \) q is _______ (true or false).
   If p is false and q is true, then p \( \land \) q is _______ (true or false).
   If p is false and q is false, then p \( \land \) q is _______ (true or false).
Chapter 3  
Section 7

If p is true and q is false, then \( p \land q \) is ___ (true or false).
If p is false and q is true, then \( p \land q \) is ___ (true or false).
If p is true and q is true, then \( p \land q \) is ___ (true or false).
If p is false and q is false, then \( p \land q \) is ___ (true or false).

D. \( p: \) "The election is tomorrow."
   \( q: \) "Mr. Jones will win."

\[ p \land q \] ___ (true or false).

If p is false and q is true, then \( p \land q \) is ___ (true or false).
If p is true and q is true, then \( p \land q \) is ___ (true or false).
If p is true and q is false, then \( p \land q \) is ___ (true or false).
If p is false and q is false, then \( p \land q \) is ___ (true or false).

TASK 19

Given two statements, p and q, form a compound statement using the logical connective or. Write a symbolic representation of the compound statement. If the truth or falsity of the two statements is known, determine the truth or falsity of the compound statement.

RULE 19

To form the compound statement, place "or" between the two statements. If the statements are represented by p and q, represent the compound statement symbolically by writing \( p \lor q \). Finally, if either or both statements are true, then the compound statement is true. If both statements are false, then the compound statement is false.

EXAMPLES 19

A. Given: \( p: \) "It is raining."
   \( q: \) "It is sunny."

Answer: \( p \lor q: \) "It is raining, or it is sunny."

Suppose \( p \) were true, then \( q \) would be false, but \( p \lor q \) would be a true statement. Suppose \( q \) were true. Then, \( p \) would be false, but \( p \lor q \) would again be true.

B. Given: \( p: \) "John is at school."
   \( q: \) "Dave is at home."

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Answer: $p \lor q$: "John is at school, or Dave is at home."

Suppose both $p$ and $q$ were true statements. Then $p \lor q$ would be a true statement. However, if they were both false, then $p \lor q$ must be false.

**EXERCISES 19**

A. $p$: "Jack is leaving."
   $q$: "Jane is arriving."

   If $p$ and $q$ are both true, then $p \lor q$ is ________ (true or false).
   If $p$ is true and $q$ is false, then $p \lor q$ is ________ (true or false).
   If $p$ is false and $q$ is true, then $p \lor q$ is ________ (true or false).
   If $p$ and $q$ are both false, then $p \lor q$ is ________ (true or false).

B. $p$: "It is thundering."
   $q$: "The wind is blowing."

   If $p$ and $q$ are both false, then $p \lor q$ is ________ (true or false).
   If $p$ and $q$ are both true, then $p \lor q$ is ________ (true or false).
   If $p$ is false and $q$ is true, then $p \lor q$ is ________ (true or false).
   If $p$ is true and $q$ is false, then $p \lor q$ is ________ (true or false).

C. $p$: "The game is Saturday."
   $q$: "Joe will play."

   If $p$ is true and $q$ is false, then $p \lor q$ is ________ (true or false).
   If $p$ and $q$ are both false, then $p \lor q$ is ________ (true or false).
   If $p$ is false and $q$ is true, then $p \lor q$ is ________ (true or false).
   If $p$ and $q$ are both true, then $p \lor q$ is ________ (true or false).

D. $p$: "You are right."
   $q$: "John is right."

   If $p$ is false and $q$ is true, then $p \lor q$ is ________ (true or false).
   If $p$ is true and $q$ is false, then $p \lor q$ is ________ (true or false).
   If $p$ and $q$ are both true, then $p \lor q$ is ________ (true or false).
   If $p$ and $q$ are both false, then $p \lor q$ is ________ (true or false).
TASK 20

Given two statements, p and q, form a conditional statement. Write a symbolic representation of the compound statement. If the truth or falsity of the two statements is known, determine the truth or falsity of the conditional statement.

RULE 20

To form the conditional, (a) write "If," (b) copy the first statement, (c) write "then" and (d) copy the second statement. If the first and second statements are represented p and q, respectively, represent the conditional by writing $p \rightarrow q$, which is read "If p, then q." Finally, if the first statement is true, and the second false, then the conditional statement is false. In all other cases the conditional is true.

EXAMPLES 20

A. Given: $p$: "John is strong."
   $q$: "John cannot lift a brick."

   Answer: $p \rightarrow q$: "If John is strong, then he cannot lift a brick."

   Suppose p is true and q is false. Then, $p \rightarrow q$ would be false. If p were false and q were true, then $p \rightarrow q$ would be true.

B. Given: $p$: "The sky is bright."
   $q$: "The air is clear."

   Answer: $p \rightarrow q$: "If the sky is bright, then the air is clear."

   Suppose p and q are both true, or both false. In either case, $p \rightarrow q$ would be true.

EXERCISES 20

A. $p$: "The wind is blowing."
   $q$: "James is cold."

   If p and q are both true, then $p \rightarrow q$ is ____________ (true or false).
   If p and q are both false, then $p \rightarrow q$ is ____________ (true or false).
   If p is true and q is false, then $p \rightarrow q$ is ____________ (true or false).
   If p is false and q is true, then $p \rightarrow q$ is ____________ (true or false).

B. $p$: "We are going to the beach."
   $q$: "The car will be full."

   If p and q are both true, then $p \rightarrow q$ is ____________ (true or false).
   If p and q are both false, then $p \rightarrow q$ is ____________ (true or false).
   If p is true and q is false, then $p \rightarrow q$ is ____________ (true or false).
   If p is false and q is true, then $p \rightarrow q$ is ____________ (true or false).
If \( p \) is true and \( q \) is false, then \( p \rightarrow q \) is \underline{true} \hspace{1cm} (true or false).
If \( p \) and \( q \) are both false, then \( p \rightarrow q \) is \underline{false} \hspace{1cm} (true or false).
If \( p \) is false and \( q \) is true, then \( p \rightarrow q \) is \underline{false} \hspace{1cm} (true or false).
If \( p \) and \( q \) are both true, then \( p \rightarrow q \) is \underline{true} \hspace{1cm} (true or false).

C. \( p: \) "The dog is lying down."
   \( q: \) "The cat is not here."
   \underline{false} \hspace{1cm} \underline{true} \hspace{1cm} \underline{false} \hspace{1cm} \underline{true} .

If \( p \) is false and \( q \) is true, then \( p \rightarrow q \) is \underline{true} \hspace{1cm} (true or false).
If \( p \) and \( q \) are both true, then \( p \rightarrow q \) is \underline{true} \hspace{1cm} (true or false).
If \( p \) and \( q \) are both false, then \( p \rightarrow q \) is \underline{true} \hspace{1cm} (true or false).
If \( p \) is true and \( q \) is false, then \( p \rightarrow q \) is \underline{true} \hspace{1cm} (true or false).

D. \( p: \) "The cow is coming home."
   \( q: \) "The time is late."
   \underline{true} \hspace{1cm} \underline{true} \hspace{1cm} \underline{true} \hspace{1cm} \underline{true} .

If \( p \) and \( q \) are both false, then \( p \rightarrow q \) is \underline{true} \hspace{1cm} (true or false).
If \( p \) is false and \( q \) is true, then \( p \rightarrow q \) is \underline{false} \hspace{1cm} (true or false).
If \( p \) and \( q \) are both true, then \( p \rightarrow q \) is \underline{true} \hspace{1cm} (true or false).
If \( p \) is true and \( q \) is false, then \( p \rightarrow q \) is \underline{false} \hspace{1cm} (true or false).

**TASK 21**

Given two statements, \( p \) and \( q \), form the compound statement "not \( p \) or \( q \)."
Write a symbolic representation of this compound statement. If the truth or falsity of the two statements is known, determine the truth or falsity of the compound statement.

**RULE 21**

Apply Rule 17 to \( p \), then apply Rule 19 to form \( \sim p \lor q \). Represent the compound statement by writing \( \sim p \lor q \). Finally, if statement \( p \) is true and statement \( q \) is false, then \( \sim p \lor q \) is false. In all other cases it is true.

**EXAMPLES 21**

Because \( \sim p \lor q \) is the same as \( p \rightarrow q \), use Examples 20, substituting \( \sim p \lor q \) for \( p \rightarrow q \).

**EXERCISES 21**

Because \( \sim p \lor q \) is the same as \( p \rightarrow q \), use Exercises 20 substituting \( \sim p \lor q \) for \( p \rightarrow q \).
TASK 22

Given the logical possibilities of a problem situation, determine the truth set for a given statement and the truth set for its negation.

RULE 22

If the statement is true for a given logical possibility (Rule 1), put that logical possibility in the truth set of the statement. If false, put that logical possibility in the truth set for the negation of the statement. Hence, to find the truth set for the negation of the statement, find the complement (Rule 5, Chapter 2) of the truth set of the statement.

EXAMPLES 22

A. Given: The set of logical possibilities for weather are {sunny, scattered clouds, rain, snow, hail}.

Answer:

The truth set for the statement "It is precipitating" is \{rain, snow, hail\}. The complement of this set \{sunny, scattered clouds\}, is precisely the truth set for the negation, "It is not precipitating."

B. Given: The set of people in the room is \{joe, Jane, John, Frank, Jill\}.

Answer:

The truth set for the statement, "That person is female" is \{jane, Jill\}. The truth set for the negation is \{joe, John, Frank\}.

EXERCISES 22

Find the truth set for each statement and its negation. Use the set of logical possibilities provided.

A. If the logical possibilities are {precipitation, no precipitation}, "The ground is dry" has as its truth set \{\}, and the negation \{\}.

B. If the logical possibilities are {black, red, brunette, blond}, "His hair color is not blond" has for its truth set \{\}, and the negation \{\}.

C. If the logical possibilities are {3 a.m., 6 a.m., 9 a.m., 12 noon, 3 p.m., 6 p.m.}, "The trains arrival time is between 8 a.m. and 5 p.m."has for its truth set \{\} and the negation \{\}.  

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SECTION 8. The Relationship Between Statement Logic and Operations on Sets

TASK 23

Given two statements and their truth sets, determine the truth set of the compound statements formed with or and and. Represent these compound statements using Venn diagrams.

RULE 23

To determine the truth set for the compound using or, find the union of the individual truth sets and give the Venn diagram for the union (Rule 14). For the compound using and, find the intersection of the truth sets and give the Venn diagram for the intersection (Rule 15).

(Take care to account for cases in which the truth sets are disjoint or one is a subset of the other.)

EXAMPLES 23

A. Given: Let {Mary, Tom, Ann, Bill, John} be the mutual acquaintances of Ed and Daphne. Then suppose "Is a close friend of Ed" and "Is a close friend of Daphne" have truth sets {Tom, Ann} and {Ann, John} respectively.

Answer:

The compound statements using and and or have truth sets {Ann} and {Tom, Ann, John} respectively (using Rules 14 and 15). The Venn diagrams are:

B. Given: Let the logical possibilities be {sun, rain, snow, hail}. The truth sets for "There is precipitation" and "There is ice on the ground" are {rain, snow, hail} and {snow, hail} respectively.

Answer:

The compound statements using and and or have truth sets {snow, hail} and {rain, snow, hail} respectively (using Rules 14 and 15). The Venn diagrams are:
EXERCISES 23

A. "Train 31 leaves after 11 a.m." and \{12 noon, 3 p.m., 6 p.m.\}.
   "Train 40 leaves before 3 p.m." and \{3 a.m., 6 a.m., 9 a.m., 12 noon\}.
B. "Train 20 leaves before 4 p.m." and \{3 a.m., 6 a.m., 9 a.m., 12 noon, 3 p.m.\}.
   "Train 19 leaves before 11 a.m." and \{3 a.m., 6 a.m., 9 a.m.\}.
C. "Train 21 leaves after 2 p.m." and \{3 p.m., 6 p.m.\}.
   "Train 18 leaves before 1 p.m." and \{3 a.m., 6 a.m., 9 a.m., 12 noon\}.

TASK 24

Given two statements \(p\) and \(q\) and their truth sets, determine the truth set of the conditional statement \((p \rightarrow q)\). Represent this truth set with a Venn diagram.

RULE 24

As seen in Task 21, \(p \rightarrow q\) is equivalent to \(\neg p \lor q\), so to find the truth set for \(p \rightarrow q\), find the truth set first for \(\neg p\) (Rule 22), then find the truth set for \(\neg p \lor q\) (Rule 23) and represent \(\neg p \lor q\) with a Venn diagram.

EXAMPLES 24

A. Given: The truth set of all logical possibilities of \(p\) and \(q\), \{1, 2, 3, 4, 5, 6, 7, 8\}, the truth set for \(p\), \{1, 2\}, and the truth set for \(q\), \{2, 3, 4\}.

Answer:

The truth set for \(p \rightarrow q\) is that of \(\neg p \lor q\). \(\neg p\) has truth set \{3, 4, 5, 6, 7, 8\} so that of \(\neg p \lor q\) is \{3, 4, 5, 6, 7, 8\} \cup \{2, 3, 4\}, that is: \{2, 3, 4, 5, 6, 7, 8\}.
The Venn diagram is:

\[ p \implies q \text{ (any area shaded)} \]

\[ \neg p \text{ covers } \neg p \text{ and } \neg \neg q \text{ covers } q \text{ so } \neg p \lor q \text{ is that area which has either } \neg \neg q \text{ or } \neg p \text{ or both.} \]

B. Given the truth set of all the logical possibilities of \( p \) and \( q \) \{a, b, c, d, e, f\}, the truth set for \( p \), \{a, b, c\}, and the truth set for \( q \), \{e, f\},

Answer:

The truth set for \( \neg p \lor q \) is \{d, e, f\} \cup \{e, f\} = \{d, e, f\}. The Venn diagram is:

\[ p \implies q \text{ (any area shaded)} \]

\[ \neg p \text{ covers } \neg p \text{ and } \neg \neg q \text{ covers } q \text{ so } \neg p \lor q \text{ is that area which has either } \neg \neg q \text{ or } \neg p \text{ or both.} \]

C. Given the truth set of all the logical possibilities of \( p \) and \( q \), \{All the months of the year\}, the truth sets for \( p \) and \( q \) are \{January, February, March\} and \{January, March\} respectively.

Answer:

The truth set for \( \neg p \lor q \) is \{April to December\} \cup \{January, March\} or \{all months but February\}. The Venn diagram is:
EXERCISES 24

Let the truth set of all the logical possibilities for p and q be the same for all three exercises, that is, \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}. Given the truth sets for p and q, find the truth sets for \(p \rightarrow q\) and the corresponding Venn diagrams.

A. The truth sets are: 
   \begin{align*}
   p & : \{1, 2, 3, 4, 5\} \\
   q & : \{3, 5\}
   \end{align*}

B. The truth sets are: 
   \begin{align*}
   p & : \{2, 3, 4, 5\} \\
   q & : \{3, 5, 9\}
   \end{align*}

C. The truth sets are: 
   \begin{align*}
   p & : \{1, 6, 9, 10\} \\
   q & : \{3, 5, 8\}
   \end{align*}

TASK 25

Given the logical possibilities of a situation and premise statements (assumed true), use a Venn diagram to determine whether or not a given conjecture is true.

RULE 25

In the same Venn diagram represent (1) the intersection of the truth sets of the premises (Rule 15) and (2) the truth set of the conjecture (potential conclusion). If the intersection of the truth sets of the premises lies entirely within (i.e., is a subset of) the truth set of the conjecture, then the conjecture is true whenever the premises are true, and, follows from them. Otherwise, the conjecture does not follow from the premises.

EXAMPLES 25

A. Given: possible weather conditions \{sun, rain, hail, snow\}, the premise p, "There is ice falling on the ground" and the conjecture q, "It is precipitating."

Answer:

The truth sets for p and q are \{hail, snow\} and \{rain, hail, snow\} respectively. Because the truth set for p is a subset of that for q, the conjecture follows from the premise. The Venn diagram is:

![Venn Diagram](image-url)
B. Suppose in Example A the premise \( p \) were, "The temperature outside is 50 degrees."

Answer:

The truth set for \( p \) is \{sun, rain\}. In this case the premise truth set is not a subset of the conjecture truth set. The conjecture does not follow from the premise.

C. Suppose in Example A the premises \( p \) were, "It is above freezing", and "It is precipitating."

Answer:

The intersection of the premise truth sets is \( \{\text{sun, rain}\} \cap \{\text{rain, hail, snow}\} = \{\text{rain}\} \). The conjecture \( q \) is, "It is raining." Because the truth set for \( p \) is a subset of (in fact equals), the truth set for \( q \) the conjecture follows from the premises.

EXERCISES 25

Let the logical possibilities be \{Tom, Joanne, Dave, Mary, Richard, Jane, Janet\}. Determine when the conjectures follow from the premises.

A. Premise: "The person's name begins with 'J'."  
Conjecture: "The person is female."
B. Premise: "The person's name does not begin with 'J'."  
Conjecture: "The person is male."
C. Premises: "The person's name does not begin with 'J'."  
"The person's name ends with 'e'."  
Conjecture: "The person is male."
SECTIONS 9 and 10. **Quantifiers Some and All** (Optional)

**TASK 26**

Given a statement, quantify it universally and existentially. Represent each
(a) symbolically and (b) iconically (with a Venn diagram).

**RULE 26**

Given a statement (which still makes sense when preceded by "all (for every)"
or "some (there exists a)"), add "For every" to the beginning of the sentence to form
the corresponding universal statement. Add "There exists a ______ such that" to
form the existential statement. (a) Symbolically, the universal quantifier "all"
and the existential quantifier "some" may be represented by "\(\forall x\)" and "\(\exists x\)" respec-
tively. (b) Iconically, "all" can be represented by a closed curve in the universe;
"some" by a single point in the closed curve.

**EXAMPLES 26**

A. Given: \(x\) is an animal.

Answer:

(a) For every \(x\), \(x\) is an animal; \(\forall x, x\) is animal.

There is an \(x\) such that \(x\) is an animal; \(\exists x, x\) is animal.

(b) Iconically

\[
\begin{align*}
\forall x, x \text{ is animal} & \quad \forall x, x \text{ is animal} \\
\exists x, x \text{ is animal} & \quad \exists x, x \text{ is animal}
\end{align*}
\]

B. Given: \(x\) is pink.

Answer:

(a) For every \(x\), \(x\) is pink; \(\forall x, x\) is pink.

There exists an \(x\) such that \(x\) is pink; \(\exists x, x\) is pink.

(b) Iconically

\[
\begin{align*}
\forall x, x \text{ is pink} & \quad \forall x, x \text{ is pink} \\
\exists x, x \text{ is pink} & \quad \exists x, x \text{ is pink}
\end{align*}
\]
EXERCISES 26

A. x is a cow.
B. x knows all.
C. x is nice.
D. x has a broken leg.

---

TASK 27

Given a statement using the quantifiers, $\forall x$ or $\exists x$, or both, form the denial of the statement.

RULE 27

(1) Place "\~" at the left of the statement.
(2) If the symbol after "\~" is "\forall x," replace "\~\forall x" by "\exists x\~." 
(3) If the symbol after "\~" is "\exists x," replace "\~\exists x" by "\forall x\~." 
(4) If the symbol following "\~" is another "\~", then delete both "\~'s". 
Continue to apply rules (2) - (4) until either (a) all quantifiers are to the left of "\~" or (b) "\~" is deleted by rule (4).

EXAMPLES 27

A. Given: $\forall x$, x is pink.
Answer:

Rule (1) gives $\~\forall x$, x is pink. Then rule (2) says to replace $\~\forall x$ by $\exists x\~$. So we have, $\exists x\~$, x is pink. Because $\exists x$, appears to the left of "\~", stop.  "$\exists x\~$, x is pink" is the negation. This is read, "There exists an x such that it is not the case that x is pink," or more briefly, "There exists an x such that x is not pink."

B. Given: $\exists x(y, x \cdot y = x$.
Answer:

Rule (1) gives $\~\exists x(y, x \cdot y = x$. By rule (3) obtain $\forall x\~y$, x \cdot y = x. Then by rule (2) obtain $\forall x\forall y$, x \cdot y = x. Because $\forall x\forall y$ appear to the left of "\~", stop. "$\forall x\forall y$, x \cdot y = x" is the negation. This is read "For all x, there exists a y such that x \cdot y \neq x."

C. Given: $\exists x(y, x \cdot y = x$.

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Answer:

Rule (1) gives $\exists x \land \forall y, x \cdot y = x$. Then by Rule (3) get $\forall x \land \exists y, x \cdot y = x$. By Rule (4) get $\forall x \land \forall y, x \cdot y = x$. Stop, because "~" has been deleted. $\forall x \land \forall y, x \cdot y = x$ is read, "For all $x$ and for all $y, x \cdot y = x$.

EXERCISES 27

A. $\exists x, x$ is blue.
B. $\forall x, \forall y, x + y = y + x$
C. $\forall x, \exists y, x + y = 0$
D. $\exists x \land \forall y, x \cdot y = 1$
CHAPTER 4
ALGEBRAIC SYSTEMS AND RELATIONSHIPS BETWEEN SYSTEMS

SECTION 1. Algebraic Systems and Concrete Embodiments

TASK 1

Given two input elements $x$ and $y$ of an embodiment (system) and a binary operation $\oplus$ defined on the embodiment (system), use a table to determine the output element $z = x \oplus y$.

RULE 1

Find the row with $x$ at its left end. Find the column with $y$ at its top. Then $z = x \oplus y$ is found at the intersection of the "x-row" and the "y-column."

EXAMPLES 1

<table>
<thead>
<tr>
<th>$\oplus$</th>
<th>$0^\circ$</th>
<th>$90^\circ$</th>
<th>$180^\circ$</th>
<th>$270^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0^\circ$</td>
<td>$0^\circ$</td>
<td>$90^\circ$</td>
<td>$180^\circ$</td>
<td>$270^\circ$</td>
</tr>
<tr>
<td>$90^\circ$</td>
<td>$90^\circ$</td>
<td>$180^\circ$</td>
<td>$270^\circ$</td>
<td>$0^\circ$</td>
</tr>
<tr>
<td>$180^\circ$</td>
<td>$180^\circ$</td>
<td>$270^\circ$</td>
<td>$0^\circ$</td>
<td>$90^\circ$</td>
</tr>
<tr>
<td>$270^\circ$</td>
<td>$270^\circ$</td>
<td>$0^\circ$</td>
<td>$90^\circ$</td>
<td>$180^\circ$</td>
</tr>
</tbody>
</table>

Given: $x = 90^\circ$, $y = 180^\circ$
Answer: $z = 90^\circ \oplus 180^\circ = 270^\circ$

Given: $x = 270^\circ$, $y = 270^\circ$
Answer: $z = 270^\circ \oplus 270^\circ = 180^\circ$
B. Table:  

<table>
<thead>
<tr>
<th>+</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>5</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

Given: \( x = 1, y = 3 \)  
Answer: \( z = 1 + 3 = 3 \)

Given: \( x = 4, y = 4 \)  
Answer: \( z = 4 + 4 = 2 \)

EXERCISES 1

A. Given table \( \oplus \) of Example 1A:  
1. \( x = 0^\circ, y = 0^\circ \)  
2. \( x = 90^\circ, y = 270^\circ \)  
3. \( x = 270^\circ, y = 180^\circ \)  
4. \( x = 180^\circ, y = 180^\circ \)  
5. \( x = 0^\circ, y = 270^\circ \)  
6. \( x = 0^\circ, y = 180^\circ \)

B. Given table + of Example 1B:  
1. \( x = 2, y = 3 \)  
2. \( x = 2, y = 4 \)  
3. \( x = 3, y = 3 \)  
4. \( x = 3, y = 4 \)  
5. \( x = 3, y = 5 \)  
6. \( x = 5, y = 5 \)

TASK 2

Given a display of a finite embodiment, two input elements \( x \) and \( y \) of the embodiment, and a binary operation, \( \oplus \), defined on the embodiment, compute the output element \( z = x \oplus y \).

RULE 2

Take the object (do the action) in the display which corresponds to \( x \); apply to \( x \).
the object (action) which corresponds to \( y \) in the manner described by the operation. The result is \( z = x \otimes y \).

EXAMPLES 2

A. Display (6 rotations of a hexagon):

Elements: \( 0^\circ, 60^\circ, 120^\circ, 180^\circ, 240^\circ, 300^\circ \) (Rotations)
Operation: followed by, denoted \( \otimes \)

Given: \( x = 60^\circ, y = 180^\circ \)
Answer: \( 60^\circ \) corresponds to the rotation \( \rightarrow 60^\circ \); applying a \( 180^\circ \) rotation to the \( 60^\circ \) rotation gives \( \rightarrow 180^\circ \). From the display then, a \( 60^\circ \) rotation followed by a \( 180^\circ \) rotation is a \( 240^\circ \) rotation. Thus, \( z = 60^\circ \otimes 180^\circ = 240^\circ \).

B. Display:

\[
\begin{array}{ccc}
\overline{0} & \overline{1} & \overline{2} \\
0 & 1 & 2 \\
3 & 4 & 5 \\
6 & 7 & 8 \\
9 & 10 & 11 \\
12 & 13 & 14 \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\overline{0} & \overline{1} & \overline{2} \\
\end{array}
\]

Elements: \( \overline{0}, \overline{1}, \overline{2} \)
Operation: "addition," denoted \( \oplus \), which is performed by selecting any number from each of the columns to be added, adding these numbers in the usual manner, and giving as answer the column of the result.

Given: \( x = \overline{2}, y = \overline{0} \)
Answer: \( 5 \) is an element in \( \overline{2} \); adding 9, which is an element of \( \overline{0} \), gives 14, an element of \( \overline{2} \). Using the display then, \( z = \overline{0} \oplus \overline{2} = \overline{2} \)
EXERCISES 2

A. Display: (5 rotations of a pentagon)

Elements: 0°, 72°, 144°, 216°, 288° (Rotations)

Operation: followed by, denoted °

Given: x = 144°, y = 216°

B. Same as Exercise 2A:

Given: x = 288°, y = 288°.

C. Display:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>13</td>
<td>14</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>17</td>
<td>18</td>
<td>19</td>
<td></td>
</tr>
</tbody>
</table>

Elements: 0, 1, 2, 3

Operation: "multiplication," denoted "\times," which is performed by selecting any number from each of the columns to be multiplied, multiplying these numbers in the usual manner, and giving as answer the column of the result.

Given: x = 2, y = 3

D. Same as Exercise 2C

Given: x = 3, y = 3
TASK 3

Given the elements and a binary operation © of a finite embodiment, construct a table which displays the inputs (pairs of elements) and outputs (single elements) of the operation.

RULE 3

Construct a two-dimensional array (i.e., table) in which each element of the embodiment is listed once along the top row and once along the left most column of the table. If x and y are elements of the embodiment where x is in the left most column and y in the top row, then the output, z = x © y, is put in the table at the intersection of the "x-row" and the "y-column."

EXAMPLES 3

A. Given the Display, Elements, and Operation of Example 2A.

Answer:

<table>
<thead>
<tr>
<th>©</th>
<th>0°</th>
<th>60°</th>
<th>120°</th>
<th>180°</th>
<th>240°</th>
<th>300°</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>0°</td>
<td>60°</td>
<td>120°</td>
<td>180°</td>
<td>240°</td>
<td>300°</td>
</tr>
<tr>
<td>60°</td>
<td>60°</td>
<td>120°</td>
<td>180°</td>
<td>240°</td>
<td>300°</td>
<td>0°</td>
</tr>
<tr>
<td>120°</td>
<td>120°</td>
<td>180°</td>
<td>240°</td>
<td>300°</td>
<td>0°</td>
<td>60°</td>
</tr>
<tr>
<td>180°</td>
<td>180°</td>
<td>240°</td>
<td>300°</td>
<td>0°</td>
<td>60°</td>
<td>120°</td>
</tr>
<tr>
<td>240°</td>
<td>240°</td>
<td>300°</td>
<td>0°</td>
<td>60°</td>
<td>120°</td>
<td>180°</td>
</tr>
<tr>
<td>300°</td>
<td>300°</td>
<td>0°</td>
<td>60°</td>
<td>120°</td>
<td>180°</td>
<td>240°</td>
</tr>
</tbody>
</table>

B. Given the Display, Elements, and Operation of Example 2B.

Answer:

<table>
<thead>
<tr>
<th>©</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

EXERCISES 3

A. Given the Display, Elements, and Operation of Exercise 2A
B. Given the Display, Elements, and Operation of Exercise 2C.
SECTION 2. Properties of Embodiments

TASK 4

Given two rotations, $r_1$ and $r_2$, in the set of rotations of a hexagon and the operation followed by $\oplus$, (see Examples 2A, 3A), determine whether or not the closure property for $\oplus$ holds for $r_1$ and $r_2$.

RULE 4

Perform rotation $r_1$ followed by rotation $r_2$. If this result, $r_3 = r_1 \oplus r_2$, is itself a rotation of a hexagon, then the closure property for $\oplus$ holds for $r_1$ and $r_2$; otherwise it does not.

EXAMPLES 4

A. Given: $r_1 = 120^\circ$, $r_2 = 300^\circ$
   
   Answer: A rotation of $120^\circ$ followed by a rotation of $300^\circ$, $(120^\circ \oplus 300^\circ)$, is itself a rotation of a hexagon ($60^\circ$). Therefore, the closure property for $\oplus$ holds for $120^\circ$ and $300^\circ$.

B. Given: $r_1 = 180^\circ$, $r_2 = 240^\circ$
   
   Answer: A rotation of $180^\circ$ followed by a rotation of $240^\circ$, $(180^\circ \oplus 240^\circ)$, is itself a rotation of a hexagon ($60^\circ$). Therefore, the closure property for $\oplus$ holds for $180^\circ$ and $240^\circ$.

EXERCISES 4

A. $r_1 = 0^\circ$, $r_2 = 60^\circ$
B. $r_1 = 120^\circ$, $r_2 = 120^\circ$
C. $r_1 = 60^\circ$, $r_2 = 300^\circ$
D. $r_1 = 300^\circ$, $r_2 = 300^\circ$

TASK 5

Given three rotations, $r_1$, $r_2$, and $r_3$, in the set of rotations of a hexagon and the operation followed by $\oplus$, (see Examples 2A, 3A), determine whether or not the associative property for $\oplus$ holds for $r_1$, $r_2$, and $r_3$. 
RULE 5

Perform rotation \( r_1 \) followed by rotation \( r_2 \). Follow the result \( r_1 \oplus r_2 \) by rotation \( r_3 \), (i.e., \( (r_1 \oplus r_2) \oplus r_3 \)). If this is the same as following rotation \( r_1 \) by the result, \( r_2 \oplus r_3 \), of rotation \( r_2 \) followed by rotation \( r_3 \) (i.e., \( r_1 \oplus (r_2 \oplus r_3) \)), then the associative property for \( \oplus \) holds for \( r_1, r_2 \) and \( r_3 \); otherwise it does not.

EXAMPLES 5

A. Given: \( r_1 = 60^\circ, r_2 = 120^\circ, r_3 = 300^\circ \)

Answer: \( (60^\circ \oplus 120^\circ) \oplus 300^\circ = 180^\circ \oplus 300^\circ = 120^\circ \)
\( 60^\circ \oplus (120^\circ \oplus 300^\circ) = 60^\circ \oplus 60^\circ = 120^\circ \)
Because \((60^\circ \oplus 120^\circ) + 300^\circ = 60^\circ \oplus (120^\circ \oplus 300^\circ)\), the associative property for \( \oplus \) holds for \( 60^\circ, 120^\circ, \) and \( 300^\circ \).

B. Given: \( r_1 = 60^\circ, r_2 = 0^\circ, r_3 = 60^\circ \)

Answer: \( (60^\circ \oplus 0^\circ) \oplus 60^\circ = 60^\circ \oplus 60^\circ = 120^\circ \)
\( 60^\circ \oplus (0^\circ \oplus 60^\circ) = 60^\circ \oplus 60^\circ = 120^\circ \)
Because \((60^\circ \oplus 0^\circ) \oplus 60^\circ = 60^\circ \oplus (0^\circ \oplus 60^\circ)\), the associative property for \( \oplus \) holds for \( 60^\circ, 0^\circ, \) and \( 60^\circ \).

EXERCISES 5

A. \( r_1 = 120^\circ, r_2 = 60^\circ, r_3 = 180^\circ \)
B. \( r_1 = 60^\circ, r_2 = 120^\circ, r_3 = 180^\circ \)
C. \( r_1 = 300^\circ, r_2 = 300^\circ, r_3 = 300^\circ \)
D. \( r_1 = 180^\circ, r_2 = 240^\circ, r_3 = 300^\circ \)

TASK 6

Given a rotation, \( r_0 \), in the set of rotations of a hexagon and the operation followed by, \( \oplus \), (see Examples 2A, 3A), determine whether or not \( r_0 \) is an identity rotation for the set of rotations of a hexagon under \( \oplus \).

RULE 6

Perform \( r_0 \oplus r \) and \( r \oplus r_0 \) for each rotation, \( r \), in the set of rotations of a hexagon. If for each \( r \), \( r_0 \oplus r = r \oplus r_0 = r \), then \( r_0 \) is an identity rotation for the set of rotations of a hexagon under \( \oplus \); otherwise it is not.
EXAMPLES 6

A. Given: \( r_0 = 0^\circ \)

Answer: Let \( r \) be, in turn, each rotation of the hexagon:

\[
\begin{align*}
 r_0 \oplus r &= \_ \quad r \oplus r_0 = \_ \\
0^\circ \oplus 0^\circ &= 0^\circ \quad 0^\circ \oplus 0^\circ = 0^\circ \\
0^\circ \oplus 60^\circ &= 60^\circ \quad 60^\circ \oplus 0^\circ = 60^\circ \\
0^\circ \oplus 120^\circ &= 120^\circ \quad 120^\circ \oplus 0^\circ = 120^\circ \\
0^\circ \oplus 180^\circ &= 180^\circ \quad 180^\circ \oplus 0^\circ = 180^\circ \\
0^\circ \oplus 240^\circ &= 240^\circ \quad 240^\circ \oplus 0^\circ = 240^\circ \\
0^\circ \oplus 300^\circ &= 300^\circ \quad 300^\circ \oplus 0^\circ = 300^\circ \\
\end{align*}
\]

Because for each \( r \), \( r_0 \oplus r = r \oplus r_0 = r \), \( 0^\circ \) is an identity rotation for the set of rotations of a hexagon under \( \oplus \).

B. Given: \( r_0 = 120^\circ \)

Answer: Let \( r \) be, in turn, each rotation of the hexagon:

\[
\begin{align*}
 r_0 \oplus r &= \_ \quad r \oplus r_0 = \_ \\
120^\circ \oplus 0^\circ &= 120^\circ \quad 0^\circ \oplus 120^\circ = 120^\circ \\
120^\circ \oplus 60^\circ &= 180^\circ \quad 60^\circ \oplus 120^\circ = 180^\circ \\
\end{align*}
\]

Because for some \( r \), \( r_0 \oplus r = r \oplus r_0 \neq r \), \( 120^\circ \) is not an identity rotation for the set of rotations of a hexagon under \( \oplus \).

EXERCISES 6

A. \( r_0 = 60^\circ \)  
B. \( r_0 = 180^\circ \)  
C. \( r_0 = 240^\circ \)  
D. \( r_0 = 360^\circ \)

---

TASK 7

Given two rotations, \( r_1 \) and \( r_2 \), in the set of rotations of a hexagon and the operation followed by, \( \oplus \) (see Examples 2A, 3A), determine whether or not the commutative property for \( \oplus \) holds for \( r_1 \) and \( r_2 \).

RULE 7

Perform rotation \( r_1 \) followed by rotation \( r_2 \) (i.e., \( r_1 \oplus r_2 \)). Perform rotation \( r_2 \)
followed by rotation \( r_1 \) (i.e., \( r_2 \oplus r_1 \)). If \( r_1 \oplus r_2 = r_2 \oplus r_1 \), then the commutative property for \( \oplus \) holds for \( r_1 \) and \( r_2 \); otherwise it does not.

**EXAMPLES 7**

A. Given: \( r_1 = 60^\circ, r_2 = 180^\circ \)
   
   Answer: \( 60^\circ \oplus 180^\circ = 240^\circ, 180^\circ \oplus 60^\circ = 240^\circ \). Because \( 60^\circ \oplus 180^\circ = 180^\circ \oplus 60^\circ \), the commutative property for \( \oplus \) holds for \( 60^\circ \) and \( 180^\circ \).

B. Given: \( r_1 = 0^\circ, r_2 = 300^\circ \)
   
   Answer: \( 0^\circ \oplus 300^\circ = 300^\circ, 300^\circ \oplus 0^\circ = 300^\circ \). Because \( 0^\circ \oplus 300^\circ = 300^\circ \oplus 0^\circ \), the commutative property for \( \oplus \) holds for \( 0^\circ \) and \( 300^\circ \).

**EXERCISES 7**

A. \( r_1 = 60^\circ, r_2 = 120^\circ \)

B. \( r_1 = 60^\circ, r_2 = 60^\circ \)

C. \( r_1 = 240^\circ, r_2 = 300^\circ \)

D. \( r_1 = 180^\circ, r_2 = 360^\circ \)

**TASK 8**

Given two rotations, \( r_1 \) and \( r_2 \), in the set of rotations of a hexagon and the operation followed by \( \oplus \), (see Examples 2A, 3A), determine whether or not \( r_1 \) and \( r_2 \) are inverses under \( \oplus \).

**RULE 8**

Perform \( r_1 \oplus r_2 \), then \( r_2 \oplus r_1 \). If \( r_1 \oplus r_2 = r_2 \oplus r_1 = r_o \), where \( r_o \) is an identity rotation (Rule 6) for \( \oplus \), then \( r_1 \) and \( r_2 \) are inverses under \( \oplus \); otherwise they are not.

**EXAMPLES 8**

A. Given: \( r_1 = 300^\circ, r_2 = 60^\circ \)
   
   Answer: \( 300^\circ \oplus 60^\circ = 0^\circ, 60^\circ \oplus 300^\circ = 0^\circ \). Because \( 300^\circ \oplus 60^\circ = 60^\circ \oplus 300^\circ = 0^\circ \), where \( 0^\circ \) is an identity rotation (Rule 6) for \( \oplus \), then \( 300^\circ \) and \( 60^\circ \) are inverses under \( \oplus \).
B. Given: \( r_1 = 300°, r_2 = 180° \)

Answer: \( 300° \oplus 180° = 120°, 180° \oplus 300° = 120° \). Because \( 300° \oplus 180° = 180° \oplus 300° = 120° \) but \( 120° \) is not an identity rotation (Examples 6B) for \( \oplus \), then \( 300° \) and \( 180° \) are not inverses under \( \oplus \).

EXERCISES 8

A. \( r_1 = 180°, r_2 = 180° \)
B. \( r_1 = 240°, r_2 = 60° \)
C. \( r_1 = 0°, r_2 = 0° \)
D. \( r_1 = 300°, r_2 = 120° \)

---

TASK 9

Given a rotation, \( r_g \), in the set of rotations of a hexagon and the operation followed by, \( \oplus \), (Examples 2A, 3A), determine whether or not \( r_g \) is a generator under \( \oplus \) of the set of rotations of a hexagon.

RULE 9

Compute: \( r_g \oplus r_g \oplus r_g \oplus r_g \oplus \ldots \oplus r_g \) is equal to either \( r_g \) or one of the previously obtained \( r \)'s (e.g., \( r_g \oplus r_g = r_g \oplus r_g \oplus r_g \oplus r_g \)). If every rotation in the set of rotations of a hexagon is obtained, then \( r_g \) is a generator, under \( \oplus \), of the set of rotations of a hexagon; otherwise it is not.

EXAMPLES 9

A. Given: \( r_g = 300° \)

Answer: Compute: \( 300°, 300° \oplus 300° = 240°, 300° \oplus 300° \oplus 300° = 180°, 300° \oplus 300° \oplus 300° \oplus 300° \oplus 300° \oplus 300° = 120°, 300° \oplus 300° \oplus 300° \oplus 300° \oplus 300° \oplus 300° = 60°, 300° \oplus 300° \oplus 300° \oplus 300° \oplus 300° \oplus 300° = 0°, 300° \oplus 300° \oplus 300° \oplus 300° \oplus 300° \oplus 300° = 300° \). Because each rotation of a hexagon is obtained, \( 300° \) is a generator, under \( \oplus \), of the set of rotations of a hexagon.

B. Given: \( r_g = 120° \)

Answer: Compute: \( 120°, 120° \oplus 120° = 240°, 120° \oplus 120° \oplus 120° = 0°, 120° \oplus 120° \oplus 120° \oplus 120° = 120° \). Because each rotation of a hexagon is not obtained (e.g., \( 60°, 180°, 300° \)), \( 120° \) is not a generator, under \( \oplus \), of the set of rotations of a hexagon.
EXERCISES 9

A. \( r_6 = 60^\circ \)
B. \( r_3 = 0^\circ \)
C. \( r_8 = 180^\circ \)
D. \( r_9 = 240^\circ \)


TASK 10

Given a rule for determining whether or not an operation in one system has a certain property (e.g., the associative property), given one or more special elements in a new system (i.e., set of elements), and an operation in the new system, generate a rule for determining whether or not the new special elements satisfy the given property under the new operation.

RULE 10

In the given rule, replace the original operation by the new operation and any special element (e.g., the identity) by its counterpart. (The result is the desired rule.)

EXAMPLES 10

A. Given Rule 4 for determining whether or not the closure property holds for two given rotations of a hexagon and the operation followed by, \( \odot \), generate a rule for determining whether or not the closure property holds for two rotations, \( r_1 \) and \( r_2 \), in the set of rotations of a pentagon under the operation "followed by", \( \odot \) (see Exercises 2A, 3A).

Answer:

Applying Rule 10 to Rule 4 gives:

Perform rotation \( r_1 \) of the pentagon followed by \( r_2 \). If this result, \( r_3 = r_1 \odot r_2 \) is itself a rotation of a pentagon, then the closure property for \( \odot \) holds for \( r_1 \) and \( r_2 \); otherwise it does not.

B. Given Rule 7 for determining whether or not the commutative property holds for two given rotations of a hexagon and operation followed by, \( \odot \), generate a rule for determining whether or not the commutative property holds for two elements \( \bar{m} \) and \( \bar{n} \) under the operation of addition, \( + \) (see Examples 2B, 3B).

Answer:

Applying Rule 10 to Rule 7 gives:
Chapter 4
Section 2

Perform the addition of elements \( m \) and \( n \) (i.e., \( m + n \)). Perform the addition of elements \( n \) and \( m \) (i.e., \( n + m \)). If \( n + m = m + n \) then the commutative property for + holds for \( m \) and \( n \); otherwise it does not.

C. Given Rule 9 for determining whether or not a given rotation in the set of rotations of a hexagon and the operation followed by, \( \circ \), is a generator of the set of rotations of a hexagon, generate a rule for determining whether or not an element \( m \) is a generator of the set \{\( 0, 1, 2, 3 \)\} under the operation multiplication, \( \circ \). (See Exercises 2C and 3B).

Answer:

Applying Rule 10 to Rule 9 gives:

Compute \( m \), \( m \circ m \), \( m \circ m \circ m \), \( m \circ m \circ m \circ m \), ... until \( m \circ m \circ m \circ m \circ m = \) equal to either \( m \) or the repetition of a previously obtained \( m \). If every element of the set \{\( 0, 1, 2, 3 \)\} is obtained, then \( m \) is a generator under \( \circ \) of the set; otherwise it is not.

EXERCISES 10

Generate (using Rules 4-9 and Rule 10) a rule for solving the given task.

A. Determine whether or not \( \bar{1}, \bar{2}, \bar{3} \) and the operation multiplication (See Exercises 2C, 3B) are associative.
B. Determine whether or not rotation \( r \) is a generator of the set of rotations of a pentagon under the operation followed by, \( \circ \), (See Exercises 2A, 3A).
C. Determine whether or not \( \bar{0}, \bar{1}, \bar{2} \) and the operation addition \( + \) (See Examples 2B, 3B) the closure property holds.
D. Determine whether or not rotation \( r_1 \) of the set of rotations of a pentagon and operation followed by, \( \circ \), (Exercises 2A, 3A) is an identity rotation.
E. Determine whether or not rotations \( r_1 \) and \( r_2 \) of the set of rotations of a pentagon are inverses under the operation followed by, \( \circ \). (See Exercise 2A, 3A).

**TASK 11**

Given a property and one or more elements and operations of a system, determine whether or not the elements satisfy the given property under the given operation.

**RULE 11**

Apply Rule 10 to the rule corresponding to the given property (Rules 4-9). Apply the derived rule to the given elements.
EXAMPLES 11

A. Determine whether or not the closure property holds for rotations 72° and 216° of the set of rotations of a pentagon and the operation followed by, o.

Answer:

Apply the rule generated by applying Rule 10 to Rule 4 (Example 10A). A rotation of 72° followed by a rotation of 216° (72° o 216°) is itself a rotation of the pentagon (i.e., 288°), so the closure property for o holds for 72° and 216°.

B. Determine whether or not the commutative property holds for 1 and 2 of Example 3B and the operation of addition; o.

Answer:

Apply the rule generated by applying Rule 10 to Rule 7 (Example 10B). 1 o 2 = 0, 2 o 1 = 0. Because 1 o 2 = 2 o 1 the commutative property for o holds for 1 and 2.

C. Determine whether or not 2 is a generator of the set {0, 1, 2, 3} under the operation o.

Answer:

Apply the rule generated by applying Rule 10 to Rule 9 (Example 10C). 2, 2 o 2 = 0, 2 o 2 o 2 = 0. Because each element of the set is not obtained (e.g., 1, 2) 2 is not a generator under o of the set {0, 1, 2, 3}.

EXERCISES 11

A. Determine whether or not for 2, 3, 3 of the set {0, 1, 2, 3} and the operation multiplication are associative. (Apply the rule derived in Exercise 10A).

B. Determine whether or not rotation 72° is a generator of the set of rotations of a pentagon under the operation followed by. (Apply the rule derived in Exercise 10B).

C. Determine whether or not for 1 and 2 of the set {0, 1, 2} and operation addition, the closure property holds. (Apply the rule derived in Exercise 10C).

D. Determine whether or not rotation 144° of the set of rotations of a pentagon is an identity rotation. (Apply the rule derived in Exercise 10D).

E. Determine whether or not rotations of 216° and 288° of the set of rotations of a pentagon are inverses under the operation followed by. (Apply the rule derived in Exercise 10E and use the result of Exercise 11 D).

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SECTION 3. **Relationships Between Systems and Embodiments**

3.1. **Generalization**

**TASK 12**

Given a three element "clock arithmetic," construct a "clock arithmetic" with n elements. (If n > 3, this is an example of generalization.) Construct a table representing the results of the operation on the embodiment.

**RULE 12**

Construct a clock arithmetic with n equal movements of $360^\circ/n$ each. Use Rule 3 to construct a table.

**EXAMPLES 12**

A. Given: $360^\circ/3 = 120^\circ$

Display:

\[
\begin{array}{cccc}
0 & 1 & 2 \\
0 & 120^\circ & 240^\circ \\
2 & 0 & 120^\circ \\
1 & 240^\circ & 0 \\
\end{array}
\]

Elements: 0, 1, 2

Operation: followed by, $\oplus$.

Table: (Applying Rule 3)

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Let n = 4

Answer:

$360^\circ/4 = 90^\circ$
Elements: 0, 1, 2, 3
Operation: followed by +.

Table: (Applying Rule 3)

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

B. Given: Same as Example 11A
Let n = 6

Answer:

\[360^\circ / 6 = 60^\circ\]
Elements: 0, 1, 2, 3, 4, 5
Operation: followed by, +.
Table: (Applying Rule 3)

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>5</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

EXERCISES 12

Given same as Example 11A:
A. Let $n = 5$
B. Let $n = 8$
C. Let $n = 12$
3.2. Isomorphisms

**TASK 13**

Given two finite embodiments, A with operation © and B with operation *, and a correspondence f between them, determine whether or not the correspondence is an isomorphism.

**RULE 13**

Check to see if f is one-to-one (Rule 7, chapter 2) and A and B have the same number of elements. Next construct table B' as follows: (1) Rearrange the elements of B (if necessary) so that each element along the top row and left most column of B' corresponds to the element in that position in table A. (2). Complete B' using Rule 3. If each element in B' corresponds by f to the element in the same position in table A, then f preserves the structure of A; otherwise it does not. If f is one-to-one, A and B have the same number of elements, and f preserves the structure of A, then f is an isomorphism; otherwise it is not.

**EXAMPLES 13**

A. Given:

<table>
<thead>
<tr>
<th>A. ©</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>B. *</th>
<th>0°</th>
<th>180°</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>0°</td>
<td>180°</td>
</tr>
<tr>
<td>180°</td>
<td>180°</td>
<td>0°</td>
</tr>
</tbody>
</table>

f. A

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

Answer:

Because f is not one-to-one (Rule 7, Chapter 2), f is not an isomorphism.

B. Given:

<table>
<thead>
<tr>
<th>A. ©</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>B. *</th>
<th>0°</th>
<th>90°</th>
<th>180°</th>
<th>270°</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>0°</td>
<td>90°</td>
<td>180°</td>
<td>270°</td>
</tr>
<tr>
<td>90°</td>
<td>90°</td>
<td>180°</td>
<td>270°</td>
<td>0°</td>
</tr>
<tr>
<td>180°</td>
<td>180°</td>
<td>270°</td>
<td>0°</td>
<td>90°</td>
</tr>
<tr>
<td>270°</td>
<td>270°</td>
<td>0°</td>
<td>90°</td>
<td>180°</td>
</tr>
</tbody>
</table>

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Chapter 4
Section 3.2

Answer:

By Rule 7 Chapter 2, \( f \) is one-to-one. Construct table \( B' \) by Rule 13:

<table>
<thead>
<tr>
<th>( B' )</th>
<th>( 0^\circ )</th>
<th>( 270^\circ )</th>
<th>( 180^\circ )</th>
<th>( 90^\circ )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0^\circ )</td>
<td>( 0^\circ )</td>
<td>( 270^\circ )</td>
<td>( 180^\circ )</td>
<td>( 90^\circ )</td>
</tr>
<tr>
<td>( 270^\circ )</td>
<td>Completing by Rule 3:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 180^\circ )</td>
<td>( 180^\circ )</td>
<td>( 90^\circ )</td>
<td>( 0^\circ )</td>
<td>( 270^\circ )</td>
</tr>
<tr>
<td>( 90^\circ )</td>
<td>( 90^\circ )</td>
<td>( 0^\circ )</td>
<td>( 270^\circ )</td>
<td>( 180^\circ )</td>
</tr>
</tbody>
</table>

Because \( f \) is one-to-one and each element in \( B' \) corresponds with the element in the same position in \( A \), i.e., \( f \) preserves the structure of \( A \), \( f \) is an isomorphism.

EXERCISES 13

A. Given: \( A \) and \( B \) of Example 13B let \( f: \overline{0} \leftrightarrow 0^\circ, \overline{1} \leftrightarrow 90^\circ, \overline{2} \leftrightarrow 180^\circ, \overline{3} \leftrightarrow 270^\circ. \)

B. Given: \( A \) and \( B \) of Example 13B, let \( f: \overline{0} \leftrightarrow 270^\circ, \overline{1} \leftrightarrow 180^\circ, \overline{2} \leftrightarrow 90^\circ, \overline{3} \leftrightarrow 0^\circ. \)

C. Given:

<table>
<thead>
<tr>
<th>A. ( \oplus )</th>
<th>( 0^\circ )</th>
<th>( 120^\circ )</th>
<th>( 240^\circ )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0^\circ )</td>
<td>( 0^\circ )</td>
<td>( 120^\circ )</td>
<td>( 240^\circ )</td>
</tr>
<tr>
<td>( 120^\circ )</td>
<td>( 120^\circ )</td>
<td>( 240^\circ )</td>
<td>( 0^\circ )</td>
</tr>
<tr>
<td>( 240^\circ )</td>
<td>( 240^\circ )</td>
<td>( 0^\circ )</td>
<td>( 120^\circ )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>B. ( \ast )</th>
<th>( 0 )</th>
<th>( 1 )</th>
<th>( 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 1 )</td>
<td>( 2 )</td>
</tr>
<tr>
<td>( 1 )</td>
<td>( 1 )</td>
<td>( 2 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( 2 )</td>
<td>( 2 )</td>
<td>( 0 )</td>
<td>( 1 )</td>
</tr>
</tbody>
</table>

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D. Given:

\[
\begin{array}{c|cc}
\text{A} & \Theta & 0^\circ & 180^\circ \\
\hline
0^\circ & 0^\circ & 180^\circ \\
180^\circ & 180^\circ & 0^\circ \\
\end{array}
\]

\[
\begin{array}{c|ccc}
\text{B} & \ast & 0^\circ & 120^\circ & 240^\circ \\
\hline
0^\circ & 0^\circ & 120^\circ & 240^\circ \\
120^\circ & 120^\circ & 240^\circ & 0^\circ \\
240^\circ & 240^\circ & 0^\circ & 120^\circ \\
\end{array}
\]

\[
\begin{align*}
f: & \quad \begin{array}{cc}
\text{A} & \text{B} \\
0^\circ & \leftrightarrow \quad 0^\circ \\
120^\circ & \leftrightarrow \quad 2 \\
240^\circ & \leftrightarrow \quad 1 \\
\end{array} \\
\end{align*}
\]
3.3. Embeddings

**TASK 14**

Given two finite embodiments, A with operation $\oplus$ and B with operation $\ast$, and a correspondence, $f$, between them, determine whether or not the correspondence is an embedding of A into B.

**RULE 14**

Consider the elements in Table B that correspond to elements in Table A. Construct Table $B'$ by applying Rule 13 to the elements in B that correspond to elements in A. If the correspondence $f$ between A and $B'$ is an isomorphism (Rule 13) and there are elements (at least one) in B that do not correspond to any element in A, then the correspondence is an embedding of A into B; otherwise it is not.

**EXAMPLES 14**

A. Given:

<table>
<thead>
<tr>
<th>A. $\oplus$</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

B. $\oplus$ | 0° | 90° | 180° | 270°
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>0°</td>
<td>90°</td>
<td>180°</td>
<td>270°</td>
</tr>
<tr>
<td>90°</td>
<td>90°</td>
<td>180°</td>
<td>270°</td>
<td>0°</td>
</tr>
<tr>
<td>180°</td>
<td>180°</td>
<td>270°</td>
<td>0°</td>
<td>90°</td>
</tr>
<tr>
<td>270°</td>
<td>270°</td>
<td>0°</td>
<td>90°</td>
<td>180°</td>
</tr>
</tbody>
</table>

$f$: \[ A \quad B \]

\[ \bar{0} \leftrightarrow 0° \]

\[ 1 \leftrightarrow 180° \]

Answer:

$0°$ and $180°$ in B correspond to elements in A. Construct $B'$ by Rule 13.

<table>
<thead>
<tr>
<th>B' $\oplus$</th>
<th>0°</th>
<th>180°</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>0°</td>
<td>180°</td>
</tr>
<tr>
<td>180°</td>
<td>180°</td>
<td>0°</td>
</tr>
</tbody>
</table>

Because by Rule 13, $f$ is an isomorphism between A and $B'$ and there are elements in B that correspond to no elements in A ($90°$, $270°$), $f$ is an embedding of A into B.
B. Given:

<table>
<thead>
<tr>
<th>A.</th>
<th>φ</th>
<th>0°</th>
<th>120°</th>
<th>240°</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>0°</td>
<td>120°</td>
<td>240°</td>
<td></td>
</tr>
<tr>
<td>120°</td>
<td>120°</td>
<td>240°</td>
<td>0°</td>
<td></td>
</tr>
<tr>
<td>240°</td>
<td>240°</td>
<td>0°</td>
<td>120°</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>B.</th>
<th>+</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

f: A        B
0° ↔ 0
120° ↔ 1
240° ↔ 2

Answer:

Because each element in B corresponds to an element in A, f is not an embedding. (Note that by Rule 13, f is an isomorphism).

EXERCISES 14

<table>
<thead>
<tr>
<th>A.</th>
<th>A.</th>
<th>φ</th>
<th>0°</th>
<th>120°</th>
<th>240°</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>0°</td>
<td>120°</td>
<td>240°</td>
<td></td>
<td></td>
</tr>
<tr>
<td>120°</td>
<td>120°</td>
<td>240°</td>
<td>0°</td>
<td></td>
<td></td>
</tr>
<tr>
<td>240°</td>
<td>240°</td>
<td>0°</td>
<td>120°</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>B.</th>
<th>φ</th>
<th>0°</th>
<th>60°</th>
<th>120°</th>
<th>180°</th>
<th>240°</th>
<th>300°</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>0°</td>
<td>60°</td>
<td>120°</td>
<td>180°</td>
<td>240°</td>
<td>300°</td>
<td></td>
</tr>
<tr>
<td>60°</td>
<td>60°</td>
<td>120°</td>
<td>180°</td>
<td>240°</td>
<td>300°</td>
<td>0°</td>
<td></td>
</tr>
<tr>
<td>120°</td>
<td>120°</td>
<td>180°</td>
<td>240°</td>
<td>300°</td>
<td>0°</td>
<td>60°</td>
<td></td>
</tr>
<tr>
<td>180°</td>
<td>180°</td>
<td>240°</td>
<td>300°</td>
<td>0°</td>
<td>60°</td>
<td>120°</td>
<td></td>
</tr>
<tr>
<td>240°</td>
<td>240°</td>
<td>300°</td>
<td>0°</td>
<td>60°</td>
<td>120°</td>
<td>180°</td>
<td></td>
</tr>
<tr>
<td>300°</td>
<td>300°</td>
<td>0°</td>
<td>60°</td>
<td>120°</td>
<td>180°</td>
<td>240°</td>
<td></td>
</tr>
</tbody>
</table>

f: A        B
0° ↔ 0°
120° ↔ 120°
240° ↔ 240°
### Chapter 4
#### Section 3.3

<table>
<thead>
<tr>
<th>B. A.</th>
<th>+</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td></td>
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<tr>
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<td>1</td>
<td>0</td>
<td></td>
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</tbody>
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<table>
<thead>
<tr>
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<th>+</th>
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<th>2</th>
<th>3</th>
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<tbody>
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<td>0</td>
<td>0</td>
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<td>2</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

\[ f: A \quad B \]
\[ 0 \leftrightarrow 0 \]
\[ 1 \leftrightarrow 2 \]

<table>
<thead>
<tr>
<th>C. A.</th>
<th>+</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
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<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>0</td>
<td></td>
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<tr>
<td>2</td>
<td>2</td>
<td>3</td>
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<td>1</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>B. B.</th>
<th>$\oplus$</th>
<th>0°</th>
<th>90°</th>
<th>180°</th>
<th>270°</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
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<tr>
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<td>270°</td>
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<td>180°</td>
<td>270°</td>
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<td>90°</td>
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</tr>
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<td>270°</td>
<td>270°</td>
<td>0°</td>
<td>90°</td>
<td>180°</td>
<td></td>
</tr>
</tbody>
</table>

\[ f: A \quad B \]
\[ 0 \leftrightarrow 0° \]
\[ 1 \leftrightarrow 90° \]
\[ 2 \leftrightarrow 180° \]
\[ 3 \leftrightarrow 270° \]

---

136
3.4. Homomorphisms (optional)

**TASK 15**

Given two finite embodiments A with operation © and B with operation *, and a correspondence f between them where f is a function (Rule 12, Chapter 2), determine whether or not the correspondence is a homomorphism.

**RULE 15**

One embodiment will have an equal or a greater number of elements involved in the correspondence than the other. Assume it is A. Apply Rule 13 (to get B') and) to determine whether f preserves the structure of A. If it does, f is a homomorphism; otherwise it is not.

**EXAMPLES 15**

A. Given.

<table>
<thead>
<tr>
<th>A. ©</th>
<th>0°</th>
<th>90°</th>
<th>180°</th>
<th>270°</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>0°</td>
<td>90°</td>
<td>180°</td>
<td>270°</td>
</tr>
<tr>
<td>90°</td>
<td>90°</td>
<td>180°</td>
<td>270°</td>
<td>0°</td>
</tr>
<tr>
<td>180°</td>
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<tr>
<td>270°</td>
<td>270°</td>
<td>0°</td>
<td>90°</td>
<td>180°</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>B. ©</th>
<th>0°</th>
<th>180°</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>0°</td>
<td>180°</td>
</tr>
<tr>
<td>180°</td>
<td>180°</td>
<td>0°</td>
</tr>
</tbody>
</table>

**Answer:**

Use Rule 13 to construct B'.

<table>
<thead>
<tr>
<th>B' ©</th>
<th>0°</th>
<th>180°</th>
<th>0°</th>
<th>180°</th>
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</thead>
<tbody>
<tr>
<td>0°</td>
<td>0°</td>
<td>180°</td>
<td>0°</td>
<td>180°</td>
</tr>
<tr>
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<td>180°</td>
<td>0°</td>
<td>0°</td>
<td>180°</td>
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<td>0°</td>
<td>0°</td>
<td>180°</td>
<td>0°</td>
<td>180°</td>
</tr>
</tbody>
</table>

137
Because in Table B each element corresponds to the element in its position in A, the correspondence is a homomorphism.

B. Given.

<table>
<thead>
<tr>
<th>A.</th>
<th>0°</th>
<th>120°</th>
<th>240°</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>0°</td>
<td>120°</td>
<td>240°</td>
</tr>
<tr>
<td>120°</td>
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<td>240°</td>
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</tr>
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<td>240°</td>
<td>240°</td>
<td>0°</td>
<td>120°</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>B.</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

\[ f: A \leftrightarrow B \]

0° ↔ 0
120° ↔ 1
240° ↔ 2

Answer:

Use Rule 13. Note that without rewriting Table B each of its elements corresponds to the element in its position in Table A. The correspondence is a homomorphism.

C. Given.

<table>
<thead>
<tr>
<th>A.</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
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<td>3</td>
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<td>1</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>B.</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
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<tr>
<td>3</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

\[ f: A \leftrightarrow B \]

0 ↔ 0
1 ↔ 1
2 ↔ 2
3 ↔ 3

Answer:

Use Rule 13. Although Table A and Table B have corresponding elements along
the top row and left column the remaining elements are not in correspondence. Therefore, $f$ does not preserve the structure of $A$ and so is not a homomorphism.

**EXERCISES 15**

**A.**

<table>
<thead>
<tr>
<th></th>
<th>$\mathbb{A}$</th>
<th>$\mathbb{B}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0^\circ$</td>
<td>$0^\circ$</td>
<td>$0$</td>
</tr>
<tr>
<td>$72^\circ$</td>
<td>$72^\circ$</td>
<td>$1$</td>
</tr>
<tr>
<td>$144^\circ$</td>
<td>$144^\circ$</td>
<td>$2$</td>
</tr>
<tr>
<td>$216^\circ$</td>
<td>$216^\circ$</td>
<td>$3$</td>
</tr>
<tr>
<td>$288^\circ$</td>
<td>$288^\circ$</td>
<td>$4$</td>
</tr>
</tbody>
</table>

$f$: $\mathbb{A}$ $\rightarrow$ $\mathbb{B}$

<table>
<thead>
<tr>
<th></th>
<th>$\mathbb{A}$</th>
<th>$\mathbb{B}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0^\circ$</td>
<td>$\rightarrow$</td>
<td>$0$</td>
</tr>
<tr>
<td>$72^\circ$</td>
<td>$\rightarrow$</td>
<td>$1$</td>
</tr>
<tr>
<td>$144^\circ$</td>
<td>$\rightarrow$</td>
<td>$2$</td>
</tr>
<tr>
<td>$216^\circ$</td>
<td>$\rightarrow$</td>
<td>$3$</td>
</tr>
<tr>
<td>$288^\circ$</td>
<td>$\rightarrow$</td>
<td>$4$</td>
</tr>
</tbody>
</table>

**B.** Same as Exercise 15A except:

$f$: $\mathbb{A}$ $\rightarrow$ $\mathbb{B}$

<table>
<thead>
<tr>
<th></th>
<th>$\mathbb{A}$</th>
<th>$\mathbb{B}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0^\circ$</td>
<td>$\rightarrow$</td>
<td>$4$</td>
</tr>
<tr>
<td>$72^\circ$</td>
<td>$\rightarrow$</td>
<td>$3$</td>
</tr>
<tr>
<td>$144^\circ$</td>
<td>$\rightarrow$</td>
<td>$2$</td>
</tr>
<tr>
<td>$216^\circ$</td>
<td>$\rightarrow$</td>
<td>$1$</td>
</tr>
<tr>
<td>$288^\circ$</td>
<td>$\rightarrow$</td>
<td>$0$</td>
</tr>
</tbody>
</table>
### Task 16

Given two finite embodiments (A (©) and B (*)), and a homomorphism, \( f \), from A to B, determine whether the homomorphism is \textit{into} or \textit{onto}.

---

#### C. A.  ©

<table>
<thead>
<tr>
<th></th>
<th>0°</th>
<th>60°</th>
<th>120°</th>
<th>180°</th>
<th>240°</th>
<th>300°</th>
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<tr>
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<td>0°</td>
<td>60°</td>
<td>120°</td>
<td>180°</td>
<td>240°</td>
</tr>
</tbody>
</table>

\[ f: \begin{array}{cc}
A & B \\
0° & 0° \\
60° & 180° \\
120° & 180° \\
180° & 0° \\
240° & 180° \\
300° & 180° \\
\end{array} \]

#### D. Same as Exercise 15C except:

\[ f: \begin{array}{cc}
A & B \\
0° & 0° \\
60° & 180° \\
120° & 180° \\
180° & 0° \\
240° & 180° \\
300° & 180° \\
\end{array} \]
RULE 16

Examine the image set of the homomorphism (those elements in \( B \) that correspond to some element in \( A \)). If the image set is equal to \( B \), the homomorphism is onto; otherwise it is into.

EXAMPLES 16

A. Given:

\[
\begin{array}{c|ccccc}
\oplus & 0^\circ & 90^\circ & 180^\circ & 270^\circ \\
\hline
0^\circ & 0^\circ & 90^\circ & 180^\circ & 270^\circ \\
90^\circ & 90^\circ & 180^\circ & 270^\circ & 0^\circ \\
180^\circ & 180^\circ & 270^\circ & 0^\circ & 90^\circ \\
270^\circ & 270^\circ & 0^\circ & 90^\circ & 180^\circ \\
\end{array}
\]

\[
\begin{array}{c|cc}
\times & 0^\circ & 180^\circ \\
\hline
0^\circ & 0^\circ & 180^\circ \\
180^\circ & 180^\circ & 0^\circ \\
\end{array}
\]

\[
f : A \rightarrow B
\]

\[
\begin{align*}
0^\circ & \leftrightarrow 0^\circ \\
90^\circ & \leftrightarrow 180^\circ \\
180^\circ & \leftrightarrow 0^\circ \\
270^\circ & \leftrightarrow 180^\circ
\end{align*}
\]

Answer:

Since each element of \( B \) is in the image set (both \( 0^\circ \) and \( 180^\circ \) correspond to elements of \( A \)), the homomorphism is onto.
B. Given:

<table>
<thead>
<tr>
<th>A. ( \oplus )</th>
<th>( 0 )</th>
<th>( 1 )</th>
<th>( 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 1 )</td>
<td>( 2 )</td>
</tr>
<tr>
<td>( 1 )</td>
<td>( 1 )</td>
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<td>( 0 )</td>
</tr>
<tr>
<td>( 2 )</td>
<td>( 2 )</td>
<td>( 0 )</td>
<td>( 1 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>B. ( \ast )</th>
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<th>( 60^\circ )</th>
<th>( 120^\circ )</th>
<th>( 180^\circ )</th>
<th>( 240^\circ )</th>
<th>( 300^\circ )</th>
</tr>
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<td>( 240^\circ )</td>
<td>( 300^\circ )</td>
</tr>
<tr>
<td>( 60^\circ )</td>
<td>( 60^\circ )</td>
<td>( 120^\circ )</td>
<td>( 180^\circ )</td>
<td>( 240^\circ )</td>
<td>( 300^\circ )</td>
<td>( 0^\circ )</td>
</tr>
<tr>
<td>( 120^\circ )</td>
<td>( 120^\circ )</td>
<td>( 180^\circ )</td>
<td>( 240^\circ )</td>
<td>( 300^\circ )</td>
<td>( 0^\circ )</td>
<td>( 60^\circ )</td>
</tr>
<tr>
<td>( 180^\circ )</td>
<td>( 180^\circ )</td>
<td>( 240^\circ )</td>
<td>( 300^\circ )</td>
<td>( 0^\circ )</td>
<td>( 60^\circ )</td>
<td>( 120^\circ )</td>
</tr>
<tr>
<td>( 240^\circ )</td>
<td>( 240^\circ )</td>
<td>( 300^\circ )</td>
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<td>( 60^\circ )</td>
<td>( 120^\circ )</td>
<td>( 180^\circ )</td>
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<tr>
<td>( 300^\circ )</td>
<td>( 300^\circ )</td>
<td>( 0^\circ )</td>
<td>( 60^\circ )</td>
<td>( 120^\circ )</td>
<td>( 180^\circ )</td>
<td>( 240^\circ )</td>
</tr>
</tbody>
</table>

\[ f: \begin{align*}
\begin{array}{c}
A \\
0 \\
1 \\
2 \\
\end{array} & \rightarrow & \begin{array}{c}
B \\
0^\circ \\
120^\circ \\
240^\circ \\
\end{array}
\end{align*} \]

Answer:

Since there are elements of B not in the image set of the homomorphism (e.g., \( 60^\circ \), \( 180^\circ \), \( 300^\circ \)), the homomorphism is into.
EXERCISES 16

Chapter 4  
Section 3.4

A.  

<table>
<thead>
<tr>
<th></th>
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<tr>
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<td>2</td>
<td>0</td>
<td>1</td>
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</table>

B.  

<table>
<thead>
<tr>
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</thead>
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</tr>
<tr>
<td>240°</td>
<td>240°</td>
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<td>120°</td>
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</tbody>
</table>

f:  

<p>| | |</p>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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</table>
### Chapter 4
#### Section 3.4

<table>
<thead>
<tr>
<th>B. A.</th>
<th>Φ</th>
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<td>120°</td>
<td>180°</td>
<td>240°</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>B.</th>
<th>Φ</th>
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<th>120°</th>
<th>240°</th>
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</thead>
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<td>0°</td>
<td>120°</td>
<td>240°</td>
</tr>
<tr>
<td>120°</td>
<td>120°</td>
<td>240°</td>
<td>0°</td>
<td></td>
</tr>
<tr>
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Chapter 4
Section 3.4

C. A. \[ \begin{array}{c|cccc} \phi & 0^\circ & 120^\circ & 240^\circ \\ \hline 0^\circ & 0^\circ & 120^\circ & 240^\circ \\ 120^\circ & 120^\circ & 240^\circ & 0^\circ \\ 240^\circ & 240^\circ & 0^\circ & 120^\circ \\ \end{array} \]

B. \[ \begin{array}{c|cccc} \phi & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & \bar{1} & \bar{2} & 3 & 4 & 5 & \bar{0} \\ 2 & \bar{2} & 3 & \bar{4} & \bar{5} & 0 & 1 \\ 3 & \bar{3} & \bar{4} & \bar{5} & 0 & 1 & 2 \\ 4 & \bar{4} & 5 & 0 & 1 & 2 & 3 \\ 5 & 5 & 0 & 1 & 2 & 3 & 4 \\ \end{array} \]

f: \[ \begin{array}{c|c} A & B \\ \hline 0^\circ & 0 \\ 120^\circ & \bar{2} \\ 240^\circ & \bar{4} \\ \end{array} \]

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PART 3
NUMBER SYSTEMS

CHAPTER 5
THE SYSTEM OF NATURAL NUMBERS

SECTIONS 1-2. Cardinality and Ordinality

TASK 1

Given a finite ordered set and an ordinal number \( n \), determine whether or not the set has the stated ordinal number.

RULE 1

Write the ordered set \([1, 2, 3, \ldots, n]\) (smallest to largest ordering) having the stated ordinal number \( n \) (e.g., write \([1, 2, 3]\) for 3, \([1, 2]\) for 2, etc.). Take each element in the given ordered set and pair it with the corresponding element (first to first, second to second, etc.) of the set constructed. Continue until all of the elements of one of the sets are used. If there are no elements remaining in the other set after the pairing, then the set has ordinal number \( n \); otherwise it does not.

EXAMPLES 1

A. Given: Ordered set \([w, x, y, z]\) (alphabetical ordering) and ordinal number 2.

Answer:

The set \([1, 2]\) is ordered (smallest to largest) and has ordinal number 2. Pair \( w \leftrightarrow 1, x \leftrightarrow 2 \). The element \( y \) can not be paired with a corresponding element of \([1, 2]\). Therefore, \([w, x, y, z]\) does not have ordinal number 2.
B. Given: Ordered set \([\{a, 5\}, \{3\}, \{8, 1, 7\}\) (number of elements in a set, smallest to largest) and ordinal number 3

Answer:

The set \([1, 2, 3]\) is ordered (smallest to largest) and has ordinal number 3. Pair \([3] \leftrightarrow 1, [a, 5] \leftrightarrow 2, [8, 1, 7] \leftrightarrow 3\). There are no elements remaining in either of the sets. Therefore \([\{a, 5\}, \{3\}, \{8, 1, 7\}\) has ordinal number 3.

EXERCISES 1

A. \([h, k, l, m, t]\) (alphabetical ordering) and ordinal number 4
B. \([l, x, 3, b]\) (left to right) and ordinal number 4
C. \([\{\frac{3}{2}, 5\}, \{3\}\) (number of elements in a set, smallest to largest) and ordinal number 2

TASK 2

Given a finite ordered set \(M\), determine its ordinal number.

RULE 2

Count the elements in the ordered set in their given order. The last number named is the ordinal number of the set.

EXAMPLES 2

A. Given: Ordered set \([b, c, a, d]\), (alphabetical ordering).

Answer:

\(a \leftrightarrow \"one,\) b \leftrightarrow \"two,\) c \leftrightarrow \"three,\) d \leftrightarrow \"four.\) The ordinal number is four.

B. Given: Ordered set \([7, 3/4, 2, 3, 6]\) (largest to smallest).

Answer:

\(7 \leftrightarrow \"one,\) 6 \leftrightarrow \"two,\) 3 \leftrightarrow \"three,\) 2 \leftrightarrow \"four,\) 3/4 \leftrightarrow \"five.\) The ordinal number is five.

EXERCISES 2

A. \([e, b, c, a, 1, f]\) (reverse alphabetical ordering)
B. \([\{5\}, \{a, 2, x\}, \{3, 7, 2, 1\}\) (number of elements in a set, largest to smallest)
C. \([x, 5]\) (right to left)
D. \{h, a, l, m\} (alphabetical ordering).

TASK 3

Given a finite ordered set and the ordinal number \(n\), determine the \(n\)th element.

RULE 3

Count the elements in the ordered set in their given order. The element corresponding to \(n\) in the counting is the \(n\)th element.

EXAMPLES 3

A. Given: Ordered set \{h, e, a, b\} (alphabetical ordering) and the ordinal number 3.

Answer:

\(a \leftrightarrow \text{"one," } b \leftrightarrow \text{"two," } e \leftrightarrow \text{"three."}\) The third element is \(e\).

B. Given: Ordered set \{3, 2, 7, 6, 1, 4, 12\} (largest to smallest), and the ordinal number 5.

Answer:

\(12 \leftrightarrow \text{"one," } 7 \leftrightarrow \text{"two," } 6 \leftrightarrow \text{"three," } 4 \leftrightarrow \text{"four," } 3 \leftrightarrow \text{"five."}\) The 5th element is 3.

EXERCISES 3

A. \{e, h, l, q, r, t\} (reverse alphabetical ordering) and the ordinal number 4
B. \{1, 7, 3, 5, 9, 12, 8\} (smallest to largest) and the ordinal number 5
C. \{7, 3, x, 4\} (right to left ordering) and the ordinal number 2
D. \{\{a\}, \{c, d, e\}, \{x, y\}, \{\}\} (number of elements in a set smallest to largest) and the ordinal number 3
SECTION 3. Definition of Cardinal and Ordinal Number

TASK 4

Given two finite ordered sets M and N, use a 1-1 correspondence to determine whether or not they have the same ordinal number.

RULE 4

"Pair" the corresponding elements of sets M and N, first to first, second to second, etc. (as in Rule 1) until all of the elements of one of the sets are used. If all elements are used (i.e., paired) in the other set, they have the same ordinal number; otherwise they do not.

EXAMPLES 4

A. Given: Ordered sets \{2, 1, 3, 4, 6\} (left to right) and \{7, 5, 3, 8, 9\} (largest to smallest).

Answer:

\[ 9 \leftrightarrow 2, 8 \leftrightarrow 1, 7 \leftrightarrow 3, 5 \leftrightarrow 4, 3 \leftrightarrow 6. \] All elements are used in \{2, 1, 3, 4, 6\}. Therefore, they have the same ordinal number.

B. Given: Ordered sets \{a, e, b, c, h\} (reverse alphabetical ordering) and \{\{3, 2\}, \{e, f, g\}, \{\Delta\}, \{3, f, \square, \Delta\}\} (number of elements in a set, smallest to largest).

Answer:

\[ h \leftrightarrow \{\Delta\}, e \leftrightarrow \{3, 2\}, c \leftrightarrow \{e, f, g\}, b \leftrightarrow \{3, f, \square, \Delta\}. \] The element \(a\) in \{a, e, b, c, h\} is unused. Therefore, they do not have the same ordinal number.

EXERCISES 4

A. \{3, 2, 7\} (left to right) and \{1, 5, 3\} (smallest to largest)
B. \{e, b, c, g, h\} (alphabetical) and \{1, 2, 4, 8, 12, 15\} (right to left)
C. \{\{1\}, \{3, 4, 5\}\} (number of elements in a set, smallest to largest) and \{a, f, g\} (left to right)
D. \{e, a, b, f, c, d\} (reverse alphabetical) and \{7, 3, 8, 12, 6, 19\} (largest to smallest)

\[\_\_\_\_\_\_\_\_. \_\_\_\_\_\_\_\_.\]

\(\bowtie\) TASK 5

Given a rule involving finite ordered sets and ordinal numbers, generate a corresponding rule involving finite unordered sets and cardinal numbers.
Write the given ordinal rule but delete all reference to order and/or ordered sets and change "ordinal" to "cardinal" wherever it appears.

**EXAMPLES 5**

A. Given: Rule 1 for determining if a given finite ordered set has ordinal number $n$. (Generate a corresponding rule for determining if a given finite unordered set has cardinal number $n$.)

Answer:

Apply Rule 5 to Rule 1: Write the set $\{1, 2, 3, \ldots, n\}$ having the stated cardinal number $n$. Take each element in the given set (one at a time) and pair it with an element of the set constructed. Continue until all of the elements of one of the sets are used. If there are no elements remaining in the other set after the pairing, then the set has cardinal number $n$; otherwise it does not.

B. Given: Rule 2 for determining the ordinal number of a given finite ordered set. (Generate a corresponding rule for determining the cardinal number of a given finite unordered set.)

Answer:

Apply Rule 5 to Rule 2. Count the elements in the given set. The last number named is the cardinal number of the set.

**EXERCISES 5**

A. Rule 4 for determining whether or not two sets have the same ordinal number (Generate a corresponding rule for determining whether or not two sets have the same cardinal number.)

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**TASK 6**

Given a task involving finite unordered sets and cardinal numbers, and a rule for finite ordered sets, apply a corresponding rule for finite unordered sets to the given task.

**RULE 6**

Apply Rule 5 to the given rule for ordered sets. Apply the derived rule to the given task.
EXAMPLES 6

A. Given: The task to determine whether or not the unordered set \( \{x, y, z\} \) has cardinal number three, and Rule 1.

Answer:

Apply Rule 5 to Rule 1 (see Example 5A): \( \{1, 2, 3\} \) has cardinal number 3. 1 \( \leftrightarrow \) x, 2 \( \leftrightarrow \) y, 3 \( \leftrightarrow \) z. There are no elements remaining in either of the sets, therefore \( \{x, y, z\} \) has cardinal number 3.

B. Given: The task to determine the cardinal number of the unordered set \( \{7, 1, 5, 3, 2\} \) and Rule 2.

Answer:

Apply Rule 5 to Rule 2 (see Example 5B): 7 \( \leftrightarrow \) "one," 1 \( \leftrightarrow \) "two," 5 \( \leftrightarrow \) "three," 3 \( \leftrightarrow \) "four," 2 \( \leftrightarrow \) "five." The cardinal number is 5.

EXERCISES 6

A. The task to determine, using 1-1 correspondence, whether or not \( \{3, 7, 2\} \) and \( \{1, 5, 9, 4\} \) have the same cardinal number, and Rule 4

B. The task to determine whether or not \( \{a, x, b, h, e, s, t\} \) has cardinal number 7, and Rule 1

C. The task to determine the cardinal number of \( \{a, f, g, k\} \) and Rule 2
SECTION 4. The Set of Natural Numbers

TASK 7

Given a finite set \( M \) and a number \( b \), show that the number of elements in a set formed by adding \( b \) elements to set \( M \) is the same as the \( b \)th successor of (i.e., counting "\( b \)-up" from) the number of elements in set \( M \).

RULE 7

Count the number of elements in the given set. Find the \( b \)th successor by counting "\( b \)-up" from the number counted. Next add \( b \) elements to the given set and count the number of elements in the set thus formed.

EXAMPLES 7

A. Given: Set \{a, c\} and 3.

Answer:

There are 2 elements in \{a, c\}. Counting "3-up" gives the 3rd successor of 2, i.e., 5. Add the elements 1, 2, 3 to \{a, c\}. The set \{a, c, 1, 2, 3\} has 5 elements.

B. Given: Set \{e, ℓ, g, \{2\}\} and 4.

Answer:

There are 4 elements in \{e, ℓ, g, \{2\}\}. Counting "4-up" gives the 4th successor of 4 -- i.e., 8. Add the elements a, b, c, d to \{e, ℓ, g, \{2\}\}. The set \{e, ℓ, g, \{2\}, a, b, c, d\} has 8 elements.

EXERCISES 7

A. \{3\} and 2
B. \{a, e, g, h\}, and 4
C. \{x, y\}, and 5
D. \{1, 7, 9, 12, 15\}, and 1
SECTION 5. Addition: Definition and Properties

TASK 8

Given two natural numbers \( m \) and \( n \), use the definition of addition (in terms of sets) to find the sum.

RULE 8

Construct a set \( M \) having \( m \) elements. Without using again any of the elements of set \( M \), construct a set \( N \) having \( n \) elements. Then construct a set containing all of the elements of \( M \) and all of the elements of \( N \) (i.e., \( M \cup N \)). Count the elements in \( M \cup N \) and call this number the sum \( m + n \).

EXAMPLES 8

A. Given: Natural numbers 2 and 3.
   Answer:
   Let \( M = \{a, b\} \) and \( N = \{x, y, z\} \). \( M \cup N = \{a, b, x, y, z\} \) and has 5 elements. Therefore, \( 2 + 3 = 5 \).

B. Given: Natural numbers 1 and 5.
   Answer:
   Let \( M = \{\} \) and \( N = \{1, 2, 3, 4, 5\} \). \( M \cup N = \{1, 2, 3, 4, 5\} \) and has 6 elements. Therefore, \( 1 + 5 = 6 \).

EXERCISES 8

A. 2 and 5
B. 1 and 4
C. 3 and 2
D. 7 and 2

   

TASK 9

Given two natural numbers \( m \) and \( n \), show that the operation of addition (in terms of sets) is well-defined (for \( m \) and \( n \)).

RULE 9

Construct two different sets, \( M \) and \( M' \), each having \( m \) elements and two different sets, \( N \) and \( N' \), each having \( n \) elements. Construct \( N \) and \( N' \) in such a way that \( M \) and \( N \) have no elements in common and \( M' \) and \( N' \) have no elements in common. Find \( M \cup N \)
and \( M' \cup N' \). Count the number of elements in each of \( M \cup N \) and \( M' \cup N' \). The number of elements in each is the same. This shows that the operation of addition (in terms of sets) is well-defined for \( m \) and \( n \).

**EXAMPLES 9**

A. Given: Natural numbers 2 and 3.

Answer:

Let \( M = \{a, b\} \), \( M' = \{x, y\} \) and \( N = \{1, 2, 3\} \), \( N' = \{c, d, e\} \). \( M \cup N = \{a, b, 1, 2, 3\} \), \( M' \cup N' = \{x, y, c, d, e\} \). They each have 5 elements; therefore the operation of addition (in terms of sets) is well-defined for 2 and 3.

B. Given: Natural numbers 5 and 3.

Answer:

Let \( M = \{a, b, c, d, e\} \), \( M' = \{x, y, z, w, t\} \) and \( N = \{m, n, q\} \), \( N' = \{a, b, c\} \). \( M \cup N = \{a, b, c, d, e, m, n, q\} \) and \( M' \cup N' = \{x, y, z, w, t, a, b, c\} \). They each have 8 elements; therefore the operation of addition (in terms of sets) is well-defined for 5 and 3.

**EXERCISES 9**

A. 2 and 5  
B. 3 and 2  
C. 4 and 5  
D. 6 and 3

**TASK 10**

Given a statement involving "addition," specify which numbers act as states and which act as operators.

**RULE 10**

A. If the statement involves "adding" more elements to a given set (one at a time), then the number of elements in the given set acts as a state, the number of elements "added" acts as an operator, and the sum acts as a state.

B. If the statement involves combining the elements in two given sets and counting them, then the number of elements in both sets act as operators and the sum acts as an operator.
EXAMPLES 10

A. Given: On Tuesday, Bill collected papers for the Boy Scouts. He told his leader that Mrs. Brown gave him 5 pounds of paper and Mr. Jones gave him 3 pounds of paper, so that he collected one, two, three, four, five, six, seven, eight pounds of paper.

Answer:

The numbers 5, 3, and 8 all act as operators.

B. Given: In calculating $3 + 4$, Johnny says "three, four, five, six, seven."

Answer:

The number 3 acts as a state, 4 acts as an operator and 7 acts as a state.

EXERCISES 10

A. Mary had five dolls and received two more on her birthday. When asked how many dolls she now had, she replied, "I had five dolls, Aunt Cora gave me one and that makes six and Uncle Frank gave me one and that makes seven."

B. On a trip Frank saw six cows in one pasture and seven horses in another. He placed a stone in a bucket for each animal he saw. At the end of the trip he was asked how many animals he saw and answered, "as many animals as there are stones in the bucket," and then proceeded to count the 13 stones.

C. Steve was showing his father the stones he found. He said, "I found three stones near the garage and five in the woods. I found one, two, three, four, five, six, seven, eight stones."

D. The Tigers Little League Team won eight games prior to July 1. During the first week of July, they won three more. The manager told them that "we had eight victories and with the last three that makes nine, ten, eleven. We now have eleven victories."

TASK 11

Given two whole numbers $m$ and $n$, determine whether or not the closure property for addition of whole numbers holds for $m$ and $n$.

RULE 11

Find $m + n$ and $n + m$. If both sums are whole numbers, then the closure property holds for addition of $m$ and $n$ (in either order); otherwise the system of whole numbers itself is not closed under addition.
EXAMPLES 11

A. Given: Whole numbers 2 and 4.

Answer:

\[ 2 + 4 = 6, \quad 4 + 2 = 6 \]

6 is a whole number, therefore the closure property for addition of whole numbers holds for 2 and 4.

B. Given: Whole numbers 0 and 3.

Answer:

\[ 0 + 3 = 3, \quad 3 + 0 = 3 \]

3 is a whole number, therefore the closure property for addition of whole numbers holds for 0 and 3.

EXERCISES 11

A. 12 and 15

B. 4 and 7

C. 2 and 5

D. 15 and 6

TASK 12

Given three whole numbers, \( l \), \( m \), and \( n \), determine whether or not the associative property for addition of whole numbers holds for \( l \), \( m \), and \( n \).

RULE 12

Find \((l + m) + n\) and then add \( n \) to this sum. Find \( m + n \) and then add this sum to \( l \). If \((l + m) + n = l + (m + n)\), then the associative property for addition of whole numbers holds for \( l \), \( m \), and \( n \); otherwise the system of whole numbers itself is not associative under addition.

EXAMPLES 12

A. Given: Whole numbers 7, 2, and 5.

Answer:

\[ 7 + 2 = 9, \quad 9 + 5 = 14, \quad 2 + 5 = 7, \quad 7 + 7 = 14 \]

\( (7 + 2) + 5 = 7 + (2 + 5) \), therefore the associative property for addition of whole numbers holds for 7, 2, and 5.

B. Given: Whole numbers, 3, 6, and 2.
Answer:

\[ 3 + 6 = 9, \quad 9 + 2 = 11. \quad 6 + 2 = 8, \quad 3 + 8 = 11. \quad (3 + 6) + 2 = 3 + (6 + 2), \]

therefore the associative property for addition of whole numbers holds for 3, 6, and 2.

EXERCISES 12

A. 14, 3, and 7
B. 27, 13, and 15
C. 7, 9, and 5
D. 12, 6, and 13

TASK 13

Given two whole numbers \( m \) and \( n \), determine whether or not the commutative property for addition of whole numbers holds for \( m \) and \( n \).

RULE 13

Find the sums \( m + n \) and \( n + m \). If \( n + m = m + n \), then the commutative property for addition of whole numbers holds for \( m \) and \( n \); otherwise the system of whole numbers itself is not commutative under addition.

EXAMPLES 13

A. Given: Whole numbers 4 and 3.

Answer:

\[ 4 + 3 = 7. \quad 3 + 4 = 7. \quad 4 + 3 = 3 + 4; \text{ therefore the commutative property for addition of whole numbers holds for 4 and 3.} \]

B. Given: Whole numbers 2 and 9.

Answer:

\[ 2 + 9 = 11. \quad 9 + 2 = 11. \quad 2 + 9 = 9 + 2; \text{ therefore the commutative property for addition of whole numbers holds for 2 and 9.} \]

EXERCISES 13

A. 15 and 27
B. 32 and 14
C. 6 and 9
D. 0 and 15
TASK 14

Given a whole number \( m \), determine whether or not 0 is an additive identity for \( m \).

RULE 14

Find the sum \( m + 0 \) and then the sum \( 0 + m \). If \( m + 0 = 0 + m = m \), then 0 is an additive identity for \( m \); if \( m + 0 \neq m \) or \( 0 + m \neq m \), then 0 is not an additive identity for the system of whole numbers.

EXAMPLES 14

A. Given: Whole number 7.

Answer:

\[ 7 + 0 = 7. \quad 0 + 7 = 7. \quad 7 + 0 = 0 + 7 = 7, \] therefore 0 is an additive identity for 7.

B. Given: Whole number 15.

Answer:

\[ 15 + 0 = 15. \quad 0 + 15 = 15. \quad 15 + 0 = 0 + 15 = 15, \] therefore 0 is an additive identity for 15.

EXERCISES 14

A. 17
B. 33
C. 56
D. 8

Note: 0 is the additive identity for all whole numbers.

TASK 15

Given two whole numbers \( m \) and \( n \), and an identity for addition of whole numbers (i.e., 0), determine whether or not \( m \) and \( n \) are inverses under addition.

RULE 15

Find \( m + n \) and \( n + m \). If \( m + n = n + m = 0 \), then \( m \) and \( n \) are inverses; otherwise they are not.
EXAMPLES 15

A. Given: Whole numbers 3 and 5.

Answer:

\[ 3 + 5 \neq 0. \] Therefore in the system of whole numbers 3 and 5 are not inverses.

B. Given: Whole numbers 37 and 5.

Answer:

\[ 37 + 5 \neq 0. \] Therefore in the system of whole numbers 37 and 5 are not inverses.

EXERCISES 15

A. 14, 2
B. 0, 0
C. 37, 0
D. 56, 5
SECTION 6. Subtraction: Definition and Properties

TASK 16

Given two whole numbers \( m \) and \( n \), where \( n \) is greater than or equal to \( m \), find the difference \( n - m \) using the definition of subtraction (in terms of sets).

RULE 16

Construct a set \( N \) containing \( n \) elements. Using only elements of \( N \), construct a set \( M \) containing \( m \) elements. Form the set \( N - M \) by taking all the elements of \( N \) which are not in \( M \). Count the elements in \( N - M \). This number represents \( n - m \).

EXAMPLES 16

A. Given: Whole numbers 3 and 7.

Answer:

Let \( N = \{x, y, z, w, e, f, g\} \). Let \( M = \{y, w, f\} \). \( N - M = \{x, z, e, g\} \) and contains four elements. Therefore, \( 7 - 3 = 4 \).

B. Given: Whole numbers 4 and 5.

Answer:

Let \( N = \{1, 2, 3, 4, 5\} \). Let \( M = \{1, 2, 3, 4\} \). \( N - M = \{5\} \) and contains 1 element. Therefore, \( 5 - 4 = 1 \).

EXERCISES 16

A. 2 and 6
B. 7 and 1
C. 5 and 4
D. 2 and 3

TASK 17

Given a statement involving "subtraction", specify which numbers act as states and which act as operators.

RULE 17

A. If the statement involves starting with a set of \( \#n \) elements and then counting down by one's \( \#m \) times, then \( n - m \) is the last number counted. In this case, \( n \) and \( n - m \) are states and \( m \) is an operator.

B. If the statement involves starting with a set of \( \#n \) elements, removing \( \#m \)
elements one at a time and then counting the remaining elements (i.e., \( n - m \)), then \( n \) acts as a state and \( m \) and \( n - m \) act as operators.

**EXAMPLES 17**

A. Given: Bill and his family are taking a car trip. Bill's father informs them that they are twelve miles from their destination. A short time later, Bill says that they have gone 5 miles and asks how many more miles they have to go. Bill's sister Jane counts eleven, ten, nine, eight, seven and says that they have seven miles to go.

Answer:

The numbers 12 and seven act as states and 5 acts as an operator.

B. Wendy has 8 oranges. She is going to share them with her three friends and gives one to Joan, one to Steven and one to Hope. When asked how many she has left, she replies, as she counts, "I have one, two, three, four, five left."

Answer:

The number eight acts as a state and the numbers three and five act as operators.

**EXERCISES 17**

A. Jack removed (one at a time) 13 books from a bookcase that holds 17 books. His teacher asked Jack how many books were left. Jack counted the remaining books and replied, "four."

B. Jimmy had 23 marbles and traded 2 of them for baseball cards. He told his friend Jane that he had 21 marbles left, since he had 23 and traded one away leaving 22 and traded another away leaving 21.

C. The ticket seller at the carnival had 340 tickets at the start of the day. After the day was over, he had five tickets left. When asked how many tickets he sold, he replied, "I had 340 tickets to begin with, there are five left, 340, 339, 338, 337, 336, 335 -- therefore there were 335 tickets sold.

D. John has a summer job cutting lawns. He takes care of the lawns for eight families. On Monday, Tuesday and Wednesday, he cut five lawns. His parents wanted to go away for the weekend and asked him how many lawns he had left to cut. John replied, "I still have to cut the Caspers' lawn, Bradys' lawn and Hendersons' lawn; I have three lawns to cut."

**TASK 18**

Given an equation of the form \( m + \square = n \), solve for \( \square \) by using the ideas of sets and 1-1 correspondence.
RULE 18

Form a set $N$ having $n$ elements and a set $M$ having $m$ elements. "Pair" the elements of $M$ and the elements of $N$. Adjoin elements to $M$ as needed to continue the pairing until all of the elements of $N$ are used. Count the number of those elements adjoined to $M$. This number is the solution.

EXAMPLES 18

A. Given: Equation $3 + \Box = 7$.

Answer:

Let $N = \{a, b, c, d, e, f, g\}$ and $M = \{7, 5, 12\}$. $7 \leftrightarrow a, 5 \leftrightarrow b, 12 \leftrightarrow c$, adjoin $x$ to $M$; $x \leftrightarrow d$, adjoin $y$ to $M$; $y \leftrightarrow e$, adjoin $z$ to $M$; $z \leftrightarrow f$, adjoin $w$ to $M$; $w \leftrightarrow g$. $M$ has been changed to $\{7, 5, 12, x, y, z, w\}$ and there were four elements adjoined to $M$. The solution is four.

B. Given: Equation $5 + \Box = 6$.

Answer:

Let $N = \{1, 2, 3, 4, 5, 6\}$ and $M = \{1, 2, 3, 4, 5\}$. $1 \leftrightarrow 1, 2 \leftrightarrow 2, 3 \leftrightarrow 3, 4 \leftrightarrow 4, 5 \leftrightarrow 5$. Adjoin $x$ to $M$; $x \leftrightarrow 6$. $M$ has been changed to $\{1, 2, 3, 4, 5, x\}$. There was one element adjoined to $M$. The solution is one.

EXERCISES 18

A. $3 + \Box = 7$
B. $4 + \Box = 5$
C. $2 + \Box = 6$
D. $12 + \Box = 12$

8) TASK 19

Given a rule for determining whether or not an operation in one system has a stated property (e.g., the associative property), a new system under another operation and/or one or more special elements of the new system, generate a rule for determining whether or not the stated property holds for given elements in the new system.

8) RULE 19

In the given rule replace the original operation by the new operation and any elements (e.g., the identity) by their counterparts. (The result is the desired rule).
EXAMPLES 19

A. Given: Rule 11 and the system of whole numbers under subtraction.

Answer:

Apply Rule 19 to Rule 11. Find \( m - n \) and \( n - m \). If both differences are whole numbers, then the closure property holds for subtraction of \( m \) and \( n \) (in either order); otherwise the system of whole numbers itself is not closed under subtraction.

B. Given: Rule 13 and the system of whole numbers under subtraction.

Answer:

Apply Rule 19 to Rule 13. Find the differences \( m - n \) and \( n - m \). If \( n - m = m - n \), then the commutative property for subtraction of whole numbers holds for \( m \) and \( n \); otherwise the system of whole numbers itself is not commutative under subtraction.

EXERCISES 19

A. Rule 12 and the whole numbers under subtraction
B. Rule 14, the whole numbers under subtraction, and 0

---

TASK 20

Given a property, and one or more elements and operations of a system, determine whether or not the elements satisfy the given property under the given operation.

RULE 20

Apply Rule 19 to the rule on addition of whole numbers corresponding to the given property (Rules 11-15). Apply the derived rule to the given elements.

EXAMPLES 20

A. Given: The closure property, whole numbers 25 and 5, and the operation of subtraction.

Answer:

Apply Rule 19 to Rule 11 (Example 19A) and then apply the derived rule to 25 and 5. \( 25 - 5 = 20 \) but \( 5 - 25 \) is not a whole number. Therefore the system of whole numbers is not closed under subtraction.
B. Given: The commutative property, whole numbers 7 and 12, and the operation of subtraction.

Answer:

Apply Rule 19 to Rule 13 (Example 19B) and then apply the derived rule to 7 and 12. 12 - 7 = 5 but 7 - 12 is not a whole number so 7 - 12 ≠ 12 - 7 and therefore the system of whole numbers does not have the commutative property under subtraction.

EXERCISES 20

A. The associative property, whole numbers, 6, 8, and 10, and the operation of subtraction
B. The associative property, whole numbers 0, 0, and 0, and the operation of subtraction
C. The identity property, whole number 6, and the operation of subtraction (use the rule generated in Exercise 19B)
D. The identity property, whole number 0, and the operation of subtraction. (Use the rule generated in Exercise 19B)
SECTION 7. Multiplication: Definition and Properties

TASK 21

Given sets \( M, N \), find the cartesian product \( M \times N \).

RULE 21

Take the first element of \( M \) and pair it with each element of \( N \) (one at a time) writing the pair as \( (m,n) \) where \( m \) is the element of \( M \) paired with the element \( n \) of \( N \). Repeat with each element of \( M \). \( M \times N \) is the set of all such pairs.

EXAMPLES 21

A. Given: \( M = \{1, 2\} \) and \( N = \{x, y\} \).
   Answer:
   \[ M \times N = \{(1, x), (1, y), (2, x), (2, y)\} \]

B. Given: \( N = \{\square, \sqrt{}\} \) and \( M = \{x, y, z, w\} \)
   Answer:
   \[ M \times N = \{(x, \square), (x, \sqrt{}), (y, \square), (y, \sqrt{}), (z, \square), (z, \sqrt{}), (w, \square), (w, \sqrt{})\} \]

EXERCISES 21

A. \( \{3, 7\} \) and \( \{1, 2\} \)
B. \( \{a\} \) and \( \{a, b, c\} \)
C. \( \{a, b\} \) and \( \{1, 2, 3\} \)
D. \( \{a, b, c\} \) and \( \{x\} \)

TASK 22

Given whole numbers \( m \) and \( n \), find the product \( m \times n \) using the definition of multiplication in terms of cartesian products of sets.

RULE 22

Construct a set \( M \) having \( m \) elements and a set \( N \) having \( n \) elements. Form the set \( M \times N \) (as in Rule 21). \( m \times n \) equals the number of elements in \( M \times N \).

EXAMPLES 22

A. Given: Whole numbers 2 and 3.
Answer:

Let \( M = \{x, y\} \) and \( N = \{1, 2, 3\} \). \( M \times N = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)\} \). There are 6 elements in \( M \times N \), therefore \( 2 \times 3 = 6 \).

B. Given: Whole numbers 4 and 2

Answer:

Let \( M = \{a, b, c, d\} \) and \( N = \{x, y\} \). \( M \times N = \{(a, x), (a, y), (b, x), (b, y), (c, x), (c, y), (d, x), (d, y)\} \). There are 8 elements in \( M \times N \), therefore \( 4 \times 2 = 8 \).

EXERCISES 22

A. 5 and 2
B. 3 and 1
C. 2 and 4
D. 4 and 3

\( \text{Task 23} \)

Given two numbers \( m, n \), and an operation defined on the numbers, show that the operation is well-defined for \( m \) and \( n \).

\( \text{Rule 23} \)

Choose two different representations of \( m \), say \( m_1, m_2 \), and two different representations of \( n \), say \( n_1, n_2 \). Perform the operation using first \( m_1 \) and \( n_1 \), and then using \( m_2 \) and \( n_2 \). If the result is the same no matter which pair of representatives is used, the operation is well-defined for \( m \) and \( n \); otherwise it is not.

\( \text{Examples 23} \)

A. Given: Natural numbers 5 and 2, and the operation of addition (in terms of sets).

Answer:

Apply Rule 8 using representations \( M_1 = \{x, y, z, w, t\}, M_2 = \{1, 2, 3, 4, 5\} \) and \( N_1 = \{3, 7\}, N_2 = \{x, p\} \). \( M_1 \cup N_1 = \{x, y, z, w, t, 3, 7\} \) and \( M_2 \cup N_2 = \{1, 2, 3, 4, 5, x, p\} \). Each set has 7 elements. The sum is the same no matter what representatives are used, therefore the operation of addition (in terms of sets) is well-defined for 5 and 2.
B. Given: Whole numbers 2 and 3, and the operation of multiplication in terms of cartesian products of sets.

Answer:

Apply Rule 22 using representations $M_1 = \{a, b\}$, $M_2 = \{1, 2\}$ and $N_1 = \{x, y, z\}$, $N_2 = \{h, k, p\}$. $M_1 \times N_1 = \{(a, x), (a, y), (a, z), (b, x), (b, y), (b, z)\}$. $M_2 \times N_2 = \{(1, h), (1, k), (1, p), (2, h), (2, k), (2, p)\}$. They each have 6 elements. The product is the same no matter what representatives are used, therefore the operation of multiplication in terms of the cartesian product of sets, is well-defined for 2 and 3.

EXERCISES 23

A. 5 and 3, and subtraction of whole numbers
B. 3 and 1, and multiplication of whole numbers
C. 4 and 3, and addition of natural numbers

TASK 24

Given two sets $M, N$, represent the cartesian product $M \times N$ as a rectangular array.

RULE 24

Write the cartesian product $M \times N$. Draw a horizontal line and a vertical line perpendicular to it. Starting at the point of intersection, move to the right along the horizontal line and mark off points corresponding to the elements of $M$. Similarly moving up along the vertical line mark off points corresponding to the elements of $N$. Mark points corresponding to each element $(m, n)$ of $M \times N$ by moving to the right to the point $m$ and then moving up along a perpendicular line to the point corresponding to $n$.

EXAMPLES 24

A. Given: Set $M = \{1, 2\}$, $N = \{c, d, e\}$.

Answer:

$M \times N = \{(1, c), (1, d), (1, e), (2, c), (2, d), (2, e)\}$. 

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B. Given: Sets $M = \{ \square, \triangle, \checkmark, \bigcirc \}$, $N = \{a, b, c\}$.

Answer:

$$M \times N = \{ (\square, a), (\square, b), (\square, c), (\triangle, a), (\triangle, b), (\triangle, c),
(\checkmark, a), (\checkmark, b), (\checkmark, c), (\bigcirc, a), (\bigcirc, b), (\bigcirc, c)\}.$$ 

\[\begin{array}{cccc}
c & . & . & . \\
b & . & . & . \\
a & . & . & . \\
\square & \triangle & \checkmark & \bigcirc
\end{array}\]

**EXERCISES 24**

A. $M = \{x, y, a\}$, $N = \{1, 2\}$
B. $M = \{x, y\}$, $N = \{c, d\}$
C. $M = \{1, 2\}$, $N = \{x\}$
D. $M = \{a, b, c\}$, $N = \{1, 2, 3, 4\}$

**TASK 25**

Given two whole numbers $m$ and $n$, determine whether or not the closure property for multiplication of whole numbers holds for $m$ and $n$.

**RULE 25**

Apply Rule 19 to Rule 11 and then apply the derived rule to $m$ and $n$.

**EXERCISES 25**

A. 5 and 3
B. 3 and 9
C. 4 and 1
D. 1 and 19


\textbf{Task 26}

Given three whole numbers, $k$, $m$, and $n$, determine whether or not the associative property for multiplication of whole numbers holds for $k$, $m$, and $n$.

\textbf{Rule 26}

Apply Rule 19 to Rule 12 and then apply the derived rule to $k$, $m$, and $n$.

\textbf{Exercises 26}

A. 6, 7, and 3 
B. 3, 6, and 9 
C. 9, 5, and 2 
D. 2, 4, and 8

\par

\textbf{Task 27}

Given two whole numbers $m$ and $n$, determine whether or not the commutative property for multiplication of whole numbers holds for $m$ and $n$.

\textbf{Rule 27}

Apply Rule 19 to Rule 13 and then apply the derived rule to $m$ and $n$.

\textbf{Exercises 27}

A. 2 and 4 
B. 4 and 8 
C. 8 and 3 
D. 9 and 1

\par

\textbf{Task 28}

Given a whole number $m$, determine whether or not 1 is a multiplicative identity for $m$.

\textbf{Rule 28}

Apply Rule 19 to Rule 14 and then apply the derived rule to $m$.

\textbf{Exercises 28}

A. 6
Note: 1 is the multiplicative identity for all whole numbers.

**Task 29**

Given two whole numbers $m$ and $n$, and an identity for multiplication of whole numbers (e.g., 1), determine whether or not $m$ and $n$ are inverses under multiplication.

**Rule 29**

Apply Rule 19 to Rule 15 and then apply the derived rule to $m$ and $n$.

**Exercises 29**

- A. 3 and 4
- B. 1 and 1
- C. 0 and 0
- D. 1 and 4
SECTION 8. Division: Definition and Properties

TASK 30

Given two natural numbers \( n \) and \( d \) with \( n \) greater than \( d \), find whole numbers \( \Box, \Delta \), satisfying \( n = (d \times \Box) + \Delta \), where \( \Delta \) is less than \( d \).

RULE 30

Divide \( n \) by \( d \). Let \( \Box \) = the quotient and \( \Delta \) = the remainder. Write

\[
n = (d \times \Box) + \Delta
\]

EXAMPLES 30

A. Given: Natural numbers 17 and 5.

Answer:

\[
5 \div 17 = 3 \text{ R} 2
\]

\[
17 = 5 \times 3 + 2
\]

B. Given: Natural numbers 43 and 19.

Answer:

\[
19 \div 43 = 2 \text{ R} 5
\]

\[
43 = 19 \times 2 + 5
\]

EXERCISES 30

A. 14 and 3
B. 71 and 15
C. 64 and 16
D. 29 and 7

\[
\text{_______} \cdot \text{_______}
\]

√TASK 31

Given two whole numbers \( m \) and \( n \), determine whether or not the closure property for division of whole numbers holds for \( m \) and \( n \).
RULE 31

Apply Rule 19 to Rule 11 and then apply the derived rule to \( m \) and \( n \).

EXERCISES 31

A. 4 and 2
B. 4 and 5
C. 1 and 1
D. 8 and 2

\[ \text{_______} \cdot \text{_______} \]

\( \sqrt{\text{TASK 32}} \)

Given three whole numbers \( k \), \( m \), and \( n \), determine whether or not the associative property for the division of whole numbers holds for \( k \), \( m \), and \( n \).

RULE 32

Apply Rule 19 to Rule 12 and then apply the derived rule to \( k \), \( m \), and \( n \).

EXERCISES 32

A. 8, 4, and 2
B. 1, 1, and 1
C. 3, 9, and 27
D. 1, 1, and 2

\[ \text{_______} \cdot \text{_______} \]

\( \sqrt{\text{TASK 33}} \)

Given two whole numbers \( m \) and \( n \), determine whether or not the commutative property for division of whole numbers holds for \( m \) and \( n \).

RULE 33

Apply Rule 19 to Rule 13 and then apply the derived rule to \( m \) and \( n \).

EXERCISES 33

A. 2 and 4
B. 1 and 1
C. 2 and 1
D. 0 and 1
Given a whole number $m$, determine whether or not 1 is a division identity for $m$.

**RULE 34**

Apply Rule 19 to Rule 14 and apply the derived rule to $m$.

**EXERCISES 34**

A. 5  
B. 1  
C. 0  
D. 17
SECTION 9. The System of Whole Numbers Under the Four Operations

**TASK 35**

Given an equation of the form \( \square \, \oplus \, m = n \), where \( m \) and \( n \) are known numbers, \( \square \) is unknown, and \( \oplus \) can be +, -, x, or +, write the corresponding equation, involving the "inverse operation" of \( \oplus \), denoted \( \ominus \), and use this equation to find \( \square \).

**RULE 35**

Write \( n \, \ominus \, m = \square \). Then compute \( n \, \ominus \, m \) by the appropriate rule.

**EXAMPLES 35**

A. Given: Equation \( \square + 3 = 5 \).
   Answer:
   The inverse operation of + is -, giving \( 5 - 3 = \square \). \( 5 - 3 = 2 \).

B. Given: Equation \( \square - 2 = 8 \).
   Answer:
   The inverse operation of - is +, giving \( 8 + 2 = \square \). \( 8 + 2 = 10 \).

**EXERCISES 35**

A. \( \square - 8 = 11 \)
B. \( \square + 7 = 13 \)
C. \( \square + 4 = 6 \)
D. \( \square \times 3 = 12 \)

---

**TASK 36**

Given three whole numbers \( \ell \), \( m \), and \( n \) determine whether or not the distributive property of multiplication over addition holds for \( \ell \), \( m \), and \( n \).

**RULE 36**

Find the sum \( m + n \) and multiply \( \ell \) by this sum. Next, find the products \( \ell \times m \) and \( \ell \times n \), and then add these products. If the answers to the preceding are equal (i.e., if \( \ell \times (m + n) = (\ell \times m) + (\ell \times n) \), then the distributive property of multiplication over addition holds for \( \ell \), \( m \), and \( n \); otherwise it does not.
EXAMPLES 36

A. Given: Whole numbers 4, 3, and 7

Answer:

\[ 3 + 7 = 10, \quad 4 \times 10 = 40. \quad 4 \times 3 = 12, \quad 4 \times 7 = 28. \quad (4 \times 3) + (4 \times 7) = 40. \]
Since \(4 \times (3 + 7) = (4 \times 3) + (4 \times 7)\), the distributive property of multiplication over addition holds for 4, 3, and 7.

B. Given: Whole numbers 5, 12 and 14.

Answer:

\[ 12 + 14 = 26, 5 \times 26 = 130. \quad 5 \times 12 = 60, \quad 5 \times 14 = 70. \quad (5 \times 12) + (5 \times 14) = 130. \]
Since \(5 \times (12 + 14) = (5 \times 12) + (5 \times 14)\), the distributive property of multiplication over addition holds for 5, 12, and 14.

EXERCISES 36

A. 4, 6 and 9
B. 13, 5 and 4
C. 8, 12 and 6
D. 15, 1 and 7

---

TASK 37

Given 3 whole numbers \(l\), \(m\), and \(n\) and viewing multiplication as repeated addition, show how the distributive property of multiplication over addition holds for \(l\), \(m\), and \(n\).

RULE 37

Write \(l \times (m + n) = (m + n) + (m + n) + ... + (m + n)\), where there are \("l"\) \((m + n)\)'s added. Rewrite this as \(l \times (m + n) = (m + m + ... + m) + (n + n + ... + n)\) and then as \(l \times (m + n) = (l \times m) + (l \times n)\).

EXAMPLES 37

A. Given: Whole numbers 4, 7, and 5.

Answer:

\[ 4 \times (7 + 5) = (7 + 5) + (7 + 5) + (7 + 5) + (7 + 5) \]
\[ 4 \times (7 + 5) = (7 + 7 + 7 + 7) + (5 + 5 + 5 + 5) \]
\[ 4 \times (7 + 5) = (4 \times 7) + (4 \times 5) \]
B. Given: Whole numbers 5, 12 and 13.

Answer:

\[ 5 \times (12 + 13) = (12 + 13) + (12 + 13) + (12 + 13) + (12 + 13) + (12 + 13) \]
\[ 5 \times (12 + 13) = (12 + 12 + 12 + 12 + 12) + (13 + 13 + 13 + 13 + 13) \]
\[ 5 \times (12 + 13) = (5 \times 12) + (5 \times 13) \]

EXERCISES 37

A. 4, 15 and 7
B. 3, 12 and 6
C. 5, 14 and 9
D. 6, 3 and 12

TASK 38

Given two whole numbers \( m \) and \( n \), where \( n \) and \( m \neq 0 \) and \( n \) divides \( m \), find the quotient \( \frac{m}{n} \), using the definition of division as repeated subtraction.

RULE 38

Subtract \( n \) from \( m \), then subtract \( n \) from the difference and repeat subtracting \( n \) from each new difference until a difference of 0 is obtained. \( m \div n \) equals the number of subtractions performed.

EXAMPLES 38

A. Given: Whole numbers 16 and 8

Answer:

\[
\begin{array}{c}
16 \\
-8 \\
-8 \\
\hline 0
\end{array}
\]

There were 2 subtractions performed. \( 16 \div 8 = 2 \).

B. Given: Whole numbers 72 and 18.
Answer:

\[
\begin{array}{c}
72 \\
-18 \\
\hline
54 \\
-18 \\
\hline
36 \\
-18 \\
\hline
18 \\
-18 \\
\hline
0
\end{array}
\]

There were 4 subtractions performed. \(72 \div 18 = 4\).

EXERCISES 38

A. 42 and 6
B. 72 and 9
C. 21 and 7
D. 685 and 137

---

TASK 39

Given four whole numbers, \(k\), \(m\), \(n\), and \(p\), show that if \(k > m\) and \(n > p\) then\n\[
(k - m) + (n - p) = (k + n) - (m + p).
\]

RULE 39

Find \((k - m)\) and \((n - p)\) and then add. Next, find \((k + n)\) and \((m + p)\) and subtract. The answers are the same so \((k - m) + (n - p) = (k + n) - (m + p)\).

EXAMPLES 39

A. Given: Whole numbers \(k = 7\), \(m = 3\), \(n = 4\), and \(p = 2\).

Answer:

\[
\begin{array}{c}
(7 - 3) = 4, \ (4 - 2) = 2, \ 4 + 2 = 6 \\
(7 + 4) = 11, \ (3 + 2) = 5, \ 11 - 5 = 6. \\
(7 - 3) + (4 - 2) = (7 + 4) - (3 + 2)
\end{array}
\]

B. Given: Whole numbers \(k = 12\), \(m = 9\), \(n = 17\), and \(p = 11\).
Answer:

\[(12 - 9) = 3,\ (17 - 11) = 6,\ 3 + 6 = 9\]
\[(12 + 17) = 29,\ (9 + 11) = 20,\ 29 - 20 = 9\]
\[(12 - 9) + (17 - 11) = (12 + 17) - (9 + 11)\]

EXERCISES 39

A. 17, 13, 12 and 5
B. 9, 4, 11 and 8
C. 72, 46, 52 and 15
D. 12, 8, 6 and 2
SECTION 10. Prime Numbers, Greatest Common Factor, and Least Common Multiple

TASK 40

Given two natural numbers $d$ and $n$, determine whether or not $d$ is a factor of $n$.

RULE 40

Apply Rule 30. If the remainder is 0, then $d$ is a factor of $n$; otherwise it is not.

EXAMPLES 40

A. Given: $n = 12, d = 4$.

Answer:

Write $12 = 4 \times 3 + 0$ by Rule 30, so 4 is a factor of 12.

B. Given: $n = 19, d = 9$.

Answer:

Write $19 = 9 \times 2 + 1$ by Rule 30. Since $\Delta = 1$, 9 is not a factor of 19.

C. Given: $n = 65, d = 13$

Answer:

Write $65 = 13 \times 5 + 0$ by Rule 30 so 13 is a factor of 65.

EXERCISES 40

A. $n = 144, d = 16$
B. $n = 81, d = 27$
C. $n = 97, d = 9$
D. $n = 54, d = 8$
E. $n = 39, d = 13$

---

TASK 41

Given a natural number $n > 1$, determine whether $n$ is a prime or a composite number.
RULE 41

Apply Rule 40 to \( n \) and \( d \) for \( d = 1, 2, 3, \ldots \), until \( d \) is a whole number \( \geq n \). If \( n \) has no factors other than itself and 1, then \( n \) is prime; otherwise \( n \) is composite. (I.e., if \( n \) has a factor \( d \) where \( 1 < d < n \) and \( n = d \times k \), then \( n \) is composite. Moreover, \( k \) and \( d \) are both factors of \( n \).)

EXAMPLES 41

A. Given: 13

Answer:

Apply Rule 40 for \( n = 13 \) and \( d = 2 \), for \( n = 13 \) and \( d = 3 \), for \( n = 13 \) and \( d = 4 \), for \( n = 13 \) and \( d = 5 \), for \( n = 13 \) and \( d = 6 \), for \( n = 13 \) and \( d = 7 \). 13 is a prime number because none of these values of \( d \) is a factor of 13.

B. Given: 35

Answer:

Apply Rule 40 for \( n = 35 \) and \( d = 2 \), for \( n = 35 \) and \( d = 3 \), for \( n = 35 \) and \( d = 4 \), for \( n = 35 \) and \( d = 5 \). Since \( d = 5 \) is a factor and \( 35 = 5 \times 7 \), 35 is composite. (5 and 7 are both factors of 35.)

EXERCISES 41

A. 17
B. 23
C. 27
D. 14
E. 45

TASK 42

Given a natural number \( k \) and factors \( m \) and \( n \), i.e., \( m \times n = k \), express this relation in a factor tree.

RULE 42

Write:

\[
\frac{k}{m, n}
\]
Note: Rule 42 can be applied repeatedly. For example, if \( m \) and \( n \) are factors of \( k \), i.e., \( m \times n = k \), and \( q \) and \( v \) are factors of \( m \), i.e., \( q \times v = m \), write:

\[
\begin{array}{c}
\text{k} \\
\text{m} \\
\text{n} \\
\text{q} \\
v
\end{array}
\]

EXAMPLES 42

A. Given: 65 and factors 13 and 5

Answer:

Factor tree:

\[
\begin{array}{c}
65 \\
13 \\
v5
\end{array}
\]

B. Given: 48 and factors 4 and 12, 4 and factors 2 and 2, and 12 and factors 3 and 4.

Answer:

Factor tree:

\[
\begin{array}{c}
48 \\
4 \\
2 \\
v3 \\
v4 \\
2 \\
v2
\end{array}
\]

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EXERCISES 42

A. 24 and factors 6 and 4
B. 36 and factors 9 and 4, 9 and factors 3 and 3, 4 and factors 2 and 2
C. 144 and factors 16 and 9, 16 and factors 2 and 8, 9 and factors 3 and 3
D. 21 and factors 7 and 3
E. 27 and factors 9 and 3, 9 and factors 3 and 3

TASK 43

Given a natural number $k$, factor it into prime numbers by using a factor tree.

RULE 43

Apply Rule 41 to determine if $k$ is a prime or if there exist factors $m$ and $n$ (i.e., $m \times n = k$). If factors $m$ and $n$ exist apply Rule 42 and represent the result in a factor tree. Apply Rules 41 and 42 (where necessary) to each of the factors $m$ and $n$, and express their factors, if any, in the factor tree. Repeat this process for each factor obtained until there are only prime factors at the end of the tree.

EXAMPLES 43

A. Given: 12

Answer:

Apply Rule 41 to get $12 = 2 \times 6$. Apply Rule 42 to express the result in the factor tree:

```
     12
    /\  \
   6  2
```

In applying Rule 41 to 2 and 6 note that 2 is prime and $6 = 2 \times 3$. Apply Rule 42 to express the result in the factor tree:

```
     12
    /\  \
   6  2
   /\  \
  2  3
```
Since 2, 2, and 3 are prime by Rule 41, stop.

B. Given: 60

Answer:

Apply Rules 41 and 42 to get the factor tree:

```
    60
     /\  \
   30 / \ 
   2 15  
    / \  \
    5 3  
```

Since 2, 5, and 3 are prime by Rule 41, stop.

EXERCISES 43

A. 15
B. 21
C. 48
D. 64
E. 144

---

TASK 44

Given two natural numbers \( m \) and \( n \), find their Greatest Common Factor (G.C.F.).

RULE 44

Use Rule 43 to form prime factor trees for both \( m \) and \( n \). Make a list of all prime factors that are common to both trees. If a prime factor is repeated more than once in both trees, include it in the list as many times as it appears in both. The G.C.F. is the product of the prime factors in this list.

EXAMPLES 44

A. Given: 10 and 12
Answer:

Use Rule 43 to construct prime factor trees.

\[
\begin{align*}
\text{10} & \quad \text{12} \\
5 & \quad 2 \\
2 & \quad 4 \\
2 & \quad 2 \\
\end{align*}
\]

2 is the only prime factor of both 10 and 12. Hence, 2 is the G.C.F.

B. Given: 24, 36

Answer:

Use Rule 43 to construct prime factor trees:

\[
\begin{align*}
\text{24} & \quad \text{36} \\
6 & \quad 9 \\
3 & \quad 3 \\
2 & \quad 2 \\
2 & \quad 2 \\
\end{align*}
\]

The common prime factors of both 24 and 36 are 2, 2, 3. 2 is included twice since it is represented twice in both factor trees. The G.C.F. is the product, \(2 \times 2 \times 3 = 12\), of these common prime factors.

EXERCISES 44

A. 25, 15
B. 27, 32
C. 144, 180
D. 65, 13
E. 13, 21
TASK 45

Given two natural numbers \( m \) and \( n \), determine whether or not they are relatively prime.

RULE 45

Apply Rule 44. If there are no prime factors common to both numbers then the G.C.F. of the two numbers is 1 and the numbers are relatively prime; otherwise they are not.

EXAMPLES 45

A. Given: 16, 15

Answer:

Applying Rule 44 gives the factor trees:

```
   16
  / \  \
 4   4
/   /\/
2   2 2 2
```

Since there are no prime factors common to both 16 and 15, the G.C.F. of 16 and 15 is 1 and so they are relatively prime.

B. Given 27, 30

Answer:

Applying Rule 44 find that the G.C.F. of 27 and 30 is 3. Hence, 27 and 30 are not relatively prime.

EXERCISES 45

A. 15, 112
B. 11, 23
C. 14, 21
D. 17, 51
E. 68, 85
TASK 46

Given a natural number \( n \), find the set of multiples of \( n \).

RULE 46

Write \( \{ n, 2n, 3n, \ldots \} \).

EXAMPLES 46

A. Given: 5
Answer:
Write \( \{ 5, 5 \times 2, 5 \times 3, 5 \times 4, \ldots \} \) or \( \{ 5, 10, 15, 20 \ldots \} \)

B. Given: 7
Answer:
Write \( \{ 7, 7 \times 2, 7 \times 3, 7 \times 4, \ldots \} \) or \( \{ 7, 14, 21, 28, \ldots \} \)

EXERCISES 46

A. 4
B. 3
C. 6
D. 8
E. 12

TASK 47

Given two natural numbers \( m \) and \( n \), find the set of numbers that are multiples of both, i.e., common multiples.

RULE 47

Apply Rule 46 to \( m \), then apply Rule 46 to \( n \). The set of common multiples is the intersection of these two sets, i.e., the set of elements common to both sets. (A natural number \( k \) is a common multiple of two or more natural numbers if each of them divides \( k \).)

EXAMPLES 47

A. Given: 3 and 2
Answer:

Apply Rule 46 to 3 to get \{3, 6, 9, 12, 15, \ldots \}. Apply Rule 46 to 2 to get \{2, 4, 6, 8, 10, 12, \ldots \}. The set of elements common to both sets is \{6, 12, 18, \ldots \}.

B. Given: 4 and 6

Answer:

Apply Rule 46 to 4 to get \{4, 8, 12, 16, 20, 24, \ldots \}. Apply Rule 46 to 6 to get \{6, 12, 18, 24, \ldots \}. The intersection set is \{12, 24, 36, \ldots \}.

EXERCISES 47

A. 2 and 4
B. 3 and 5
C. 4 and 5
D. 6 and 7
E. 2 and 6

TASK 48

Given two natural numbers \(m\) and \(n\), find the Least Common Multiple (L.C.M.).

RULE 48

(1) Apply Rule 47 to find the set of common multiples of \(m\) and \(n\). The L.C.M is the least element in this set.

(2) Apply Rule 43 to \(m\) and \(n\). Make a list of all prime factors of \(m\) including repetitions. Add to this list all of the prime factors of \(n\) that are not included (including repetitions). The L.C.M. is the product of the prime factors in this list.

EXAMPLES 48

A.1. Given: 2 and 3

Answer:

Apply Rule 48(1). This gives the set \{6, 12, 18, 24 \ldots \}. The L.C.M. is 6, the least element in this set.

A.2. Given: 2 and 3
Apply Rule 48(2). 2 and 3 are prime numbers so the list of prime factors of 2 and 3 is \{2, 3\} and the L.C.M. is \(2 \times 3 = 6\).

B. 1. Given: 20 and 18

Answer:

Apply Rule 48(1). This gives the set \{180, 360, 540, 720, \ldots\}. The L.C.M. is 180, the least element in this set.

B. 2. Given: 20 and 18

Answer:

Apply Rule 48(2) to 20 and 18 giving the factor trees:

\[
\begin{align*}
4 & \quad 20 \\
6 & \quad 5 \\
2 & \quad 2 \\
& \quad 9 \\
& \quad 18 \\
& \quad 3 \\
& \quad 3
\end{align*}
\]

The prime factors of 20 are 2, 2, 5. To this add the "extra" prime factors of 18, giving 2, 2, 5, 3, 3. Hence the L.C.M. is \(2 \times 2 \times 5 \times 3 \times 3 = 180\).

EXERCISES 48

A. 4, 8
B. 12, 18
C. 24, 18
D. 8, 12
E. 36, 20
SECTION 1. Concrete and Iconic Names

TASK 1

Represent the number of elements in a given set without using numerals.

RULE 1

Choose one object (e.g., toothpicks, marbles, marks on paper) for each object in the original set.

EXAMPLES 1.

A. Given: \{a, f, g, l, q\}.

Answer:

\[ \blacksquare \blacksquare \blacksquare \blacksquare \blacksquare \]

B. Given: \{1, 7, 2, 4, 3, 6\}

Answer:

\[ \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \]

EXERCISES 1

A. \{a, f, g\}
B. \{3, 7, 9, 10, 11\}
C. \{o, p, r, q\}
D. \{1, 2, 3\}
SECTION 2. **Symbolic Names**

**TASK 2**

Given a number \( n \), represented as a numeral in base ten, represent it as a numeral in base twelve.

**RULE 2**

(In the following write \( T \) for ten and \( E \) for eleven, wherever these numbers arise.)

Find a whole number \( j = 1, 2, 3, \ldots \) such that \( 12^j \cdot T \) is greater than \( n \) and \( 12^j \) is less than or equal to \( n \). Divide \( n \) by \( 12^j \) and write the quotient in base twelve.

Next divide the remainder by \( 12^{j-1} \) and write the quotient to the right of the last quotient. Divide the second remainder by \( 12^{j-2} \) and write the quotient to the right of the last quotient. Continue this process until you divide the last remainder by 12. Finally, write this quotient to the right of the last quotient and then the remainder itself to the right of all the quotients.

**EXAMPLES 2**

A. Given: 1134

Answer:

\[
12^1 = 12, \ \ 12^2 = 144, \ \ 12^3 = 1728. \quad \text{Thus } j = 2.
\]

\[
1134 ÷ 144 = 7 \text{ with a remainder of 126.}
\]

Write 7,

\[
126 ÷ 12 = 10 = T \text{ with a remainder of 6.}
\]

Write 7T6.

B. Given: 2231

Answer:

\[
12^1 = 12, \ 12^2 = 144, \ 12^3 = 1728, \ 12^4 = 20736. \quad \text{Thus } j = 3
\]

\[
2231 ÷ 1728 = 1 \text{ with a remainder of 503.}
\]

Write 1,

\[
503 ÷ 144 = 3 \text{ with a remainder of 71.}
\]

Write 13,

\[
71 ÷ 12 = 5 \text{ with a remainder of 11 (=}E).}
\]

Write 135E.

**EXERCISES 2**

A. 573
B. 1531
C. 214
D. 3241
TASK 3

Given any digit of a numeral in base $b$, state the numerical value of the digit (in base ten).

RULE 3

Count the number of places ($p$) from the right that the digit appears. Multiply the digit by $b^k$ where $k = p - 1$. If the digit appears in the first place on the right, write the digit itself.

EXAMPLES 3

A. Given: 7201 in the base 8 numeration system, state the number represented by "7."

Answer:

The "7" appears four places from the right. Multiply 7 by $8^3$, i.e., $7 \times 8 \times 8 \times 8$. The number represented by 7 is 3,584.

B. Given: 4312 in the base 12 numeration system, state the number represented by "3."

Answer:

The "3" appears three places from the right. Multiply 3 by $12^2$ i.e., $3 \times 12 \times 12$. The number represented by 3 is 432.

EXERCISES 3

A. Given 537 (base ten), state the number represented by "3."

B. Given 3E9 (base twelve), state the number represented by "E."

C. Given 21022 (base three), state the number represented by "1."

D. Given 5342 (base six), state the number represented by "5."

TASK 4

A. Given a rule stated in terms of a variable $n$, and a value $k$ of that variable, generate a similar rule involving the given value $k$.

B. Given a rule stated in terms of a constant element (or operation) $k_1$, and a new value $k_2$, generate a similar rule involving the (given) different value $k_2$. 
RULE 4

A. Replace each occurrence of \( n \) in the old rule by \( k \).
B. Replace each occurrence of \( k^2 \) in the old rule by \( k_2 \).

EXAMPLES 4

A. Given: Rule 2 and base 2.
Answer:

Apply Rule 4 to Rule 2, replacing each occurrence of 12 by 2. This gives:
Find a whole number \( j = 1, 2, 3, \ldots \) such that \( 2^{j+1} \) is greater than \( n \) and \( 2^j \) is less than or equal to \( n \). Divide \( n \) by \( 2^j \) and write the quotient in base 2. Divide the remainder by \( 2^{j-1} \) and write the quotient to the right of the last quotient. Divide the second remainder by \( 2^{j-2} \) and write the quotient to the right of the last quotient. Continue this process until you divide the last remainder by 2. Finally, write this quotient to the right of the last quotient and then the remainder itself to the right of all the quotients.

B. Given: Rule 2 and base 5.
Answer:

Apply Rule 4 to Rule 2, replacing each occurrence of 12 by 5. This gives:
Find a whole number \( j = 1, 2, 3, \ldots \) such that \( 5^{j+1} \) is greater than \( n \) and \( 5^j \) is less than or equal to \( n \). Divide \( n \) by \( 5^j \) and write the quotient in base 5. Divide the remainder by \( 5^{j-1} \) and write the quotient to the right of the last quotient. Divide the second remainder by \( 5^{j-2} \) and write the quotient to the right of the last quotient. Continue this process until you divide the last remainder by 5. Finally, write this quotient to the right of the last quotient and then the remainder itself to the right of all the quotients.

C. Given: The rule, \( n/2 \times (1 + n) \), for finding the sum of the arithmetic series \( 1 + 2 + 3 + \ldots + n \), and the arithmetic series \( 1 + 2 + 3 + \ldots + 19 \).
Answer:

Apply Rule 4 to the above rule, replacing each occurrence of \( n \) by 19. This gives: \( 19/2 \times (1 + 19) \)

EXERCISES 4

A. Rule 2 and base 7
B. The rule \( n/2 \times (1 + n) \) and the series \( 1 + 2 + 3 + \ldots + 42 \)
C. Rule 2 and base 6
D. Rule 2 and base 3
E. Rule 2 and base 8
\textbf{TASK 5}

Given a rule stated in terms of a variable \( n \) (or constant \( k \)), and a task that requires using a similar rule involving a (new) value \( k \) of that variable (or \( k^2 \) of that constant), solve the given task.

\textbf{RULE 5}

Apply Rule 4 to the given rule in \( n \) (or \( k \)). Apply the derived rule in \( k \) (or \( k^2 \)) to the task.

\textbf{EXAMPLES 5}

A. Given: Rule 2 and the task of changing \( 215_{10} \) to a base two numeral.

Answer:

Apply the rule derived by Rule 4 in Example 4A: \( 2^1 = 2, 2^2 = 4, 2^3 = 8, 2^4 = 16, 2^5 = 32, 2^6 = 64, 2^7 = 128, 2^8 = 256 \). Thus \( j = 7 \).

\begin{align*}
215 & \div 128 = 1 \quad \text{with remainder 87} \\
87 & \div 64 = 1 \quad \text{with remainder 23} \\
23 & \div 32 = 0 \quad \text{with remainder 23} \\
23 & \div 16 = 1 \quad \text{with remainder 7} \\
7 & \div 8 = 0 \quad \text{with remainder 7} \\
7 & \div 4 = 1 \quad \text{with remainder 3} \\
3 & \div 2 = 1 \quad \text{with remainder 1} \\
\text{write 11010111}
\end{align*}

B. Given: Rule 2 and the task of changing \( 437_{10} \) to a base five numeral.

Answer:

Apply the rule derived by Rule 4 in Example 4B: \( 5^1 = 5, 5^2 = 25, 5^3 = 125, 5^4 = 625 \). Thus \( j = 3 \).

\begin{align*}
437 & \div 125 = 3 \quad \text{with remainder 62} \\
62 & \div 25 = 2 \quad \text{with remainder 12} \\
12 & \div 5 = 2 \quad \text{with remainder 2} \\
\text{write 32222}
\end{align*}
C. Given: The rule \( \frac{n}{2} \times (1 + n) \) and the task of finding the sum of the series 
\( 1 + 2 + 3 + \ldots + 19 \).

Answer:

The rule derived by Rule 4 in Example 4C is \( \frac{19}{2} \times (1 + 19) \). The sum 
\( 1 + 2 + 3 + \ldots + 19 \) is \( \frac{19}{2} \times 20 = 190 \).

EXERCISES 5

A. Rule 2 and the task of changing 649 to a base five numeral.

B. The rule \( \frac{n}{2} \times (1 + n) \) and the task of finding the sum of the series 
\( 1 + 2 + 3 + \ldots + 32 \).

C. Rule 2 and the task of changing 9710 to a base six numeral.

D. Rule 2 and the task of changing 47810 to a base three numeral.

---

TASK 6

Given a number represented as a base \( b \) numeral, represent it as a base 10 numeral.

RULE 6

Apply Rule 3 to each digit of the base \( b \) numeral. Then add the results to obtain the base 10 numeral.

EXAMPLES 6

A. Given: 1452 (base six).

Answer:

\[
\begin{align*}
1 & \text{ represents } 1 \times 6 \times 6 \times 6 = 216 \\
4 & \text{ represents } 4 \times 6 \times 6 = 144 \\
5 & \text{ represents } 5 \times 6 = 30 \\
2 & \text{ represents } 2 \\
\hline
& = 392
\end{align*}
\]

The sum is 392. Thus 392 (base 10) corresponds to 1452 (base 6).

B. Given: TBE5 (base twelve)
Answer:

\[ T \text{ represents } 10 \times 12 \times 12 \times 12 = 17280 \]
\[ 8 \text{ represents } 8 \times 12 \times 12 = 1152 \]
\[ E \text{ represents } 11 \times 12 = 132 \]
\[ 5 \text{ represents } 5 = 5 \]

\[ 18569 \]

The sum is 18569. Thus 18569 (base 10) corresponds to TBE5 (base 12).

EXERCISES 6

A. 4216 (base 7)
B. 102210 (base 3)
C. 241 (base 11)
D. 3056 (base 9)

TASK 7

Given \( n \) inches, represent it in terms of the mixed base involving yards, feet, and inches.

RULE 7

Divide \( n \) by 36. Call the quotient \( q_1 \) and, the remainder \( r_1 \). Divide \( r_1 \) by 12. Call the quotient \( q_2 \) and, the remainder \( r_2 \). \( n \) inches can be represented as \( q_1 \) yards, \( q_2 \) feet, \( r_2 \) inches.

EXAMPLES 7

A. Given: 157 inches.

Answer:

\[ 157 \div 36 = 4 \text{ with a remainder of } 13. \]
\[ 13 \div 12 = 1 \text{ with a remainder of } 1. \]

157 inches can be represented as 4 yards, 1 foot, 1 inch.

B. Given: 437 inches.

Answer:

\[ 437 \div 36 = 12 \text{ with a remainder of } 5. \]
5 + 12 = 0 with a remainder of 5.

437 inches can be represented as 12 yards, 0 feet, 5 inches.

EXERCISES 7

A. 256 inches
B. 168 inches
C. 32 inches
D. 576 inches
SECTION 3. Addition Algorithm

TASK 8

Given a pair of two-digit numbers mn and pq \((mn + pq < 100)\), justify the addition algorithm by constructing a concrete representation.

RULE 8

Represent the \(m\) tens in \(mn\) by \(m\) groups of ten (bundled) tally marks (e.g., \(####\)) and the \(n\) units in \(mn\) by \(n\) individual tally marks (e.g., \(/\)/). Similarly, represent \(p\) in \(pq\) by \(p\) groups of ten tally marks and \(q\) in \(pq\) by \(q\) individual tally marks. Then, gather all individual tally marks together. Count the number of individual tally marks. If there are ten or more, group them by tens (i.e., put them in bundles of ten). Next, gather these bundles (of ten) with the others. Count the total number of groups of ten (say \(r\)) and the total number of tally marks remaining (say \(s\)). The sum is \(rs\).

EXAMPLES 8

A. Given: 46 and 37

Answer:

46 can be represented

\[\begin{array}{c}
\text{\(####\)} \\
\text{\(##\)}
\end{array}\]

37 can be represented

\[\begin{array}{c}
\text{\(##\)} \\
\text{\(###\)}
\end{array}\]

After regrouping the tally marks (as shown on the right), there are 8 groups of ten and 3 tally marks.

Hence, \(46 + 37 = 83\).
B. Given: 27 and 48

Answer:

27 can be represented

48 can be represented

After regrouping the individual tally marks (as shown on the right), there are 7 groups of ten and 5 tally marks.

Hence, 27 + 48 = 75.

EXERCISES 8

A. 25 and 34
B. 47 and 35
C. 19 and 75
D. 53 and 38

TASK 9 (Optional)

Given an algebraic justification of the addition algorithm, supply the reason for each step in the justification.

RULE 9

A. If a pair of parentheses is shifted in a step so that all numbers remain the same and in same relative position, then the reason for this step is the associative property.
B. If the order of a pair of numbers and/or expressions in parentheses is switched, then the reason for this step is the commutative property.
C. If a number (or expression in parentheses) is written in a different form (e.g., an addition is performed, or say 7 is written as 3 + 4), then the reason for this step is renaming.
EXAMPLES 9

A. Given: 23 + 49 = 72 because:

\[
23 + 49 = (20 + 3) + (40 + 9)
= 20 + (3 + (40 + 9))
= 20 + ((40 + 9) + 3)
= 20 + (40 + (9 + 3))
= (20 + 40) + (9 + 3)
= 60 + (10 + 2)
= (60 + 10) + 2
= 70 + 2
= 72
\]

B. Given: 37 + 56 = 93 because:

\[
37 + 56 = (30 + 7) + (50 + 6)
= 30 + (7 + (50 + 6))
= 30 + ((50 + 6) + 7)
= 30 + (50 + (6 + 7))
= (30 + 50) + (6 + 7)
= 80 + (10 + 3)
= (80 + 10) + 3
= 90 + 3
= 93
\]

EXERCISES 9

A. Given: 57 + 22 = 79 because:

\[
57 + 22 = (50 + 7) + (20 + 2)
= 50 + (7 + (20 + 2))
= 50 + ((20 + 2) + 7)
= 50 + (20 + (2 + 7))
= (50 + 20) + (2 + 7)
= 70 + 9
= 79
\]

B. Given: 27 + 42 = 69 because

\[
27 + 42 = (20 + 7) + (40 + 2)
= 20 + (7 + (40 + 2))
= 20 + ((40 + 2) + 7)
= 20 + (40 + (2 + 7))
= (20 + 40) + (2 + 7)
= 60 + 9
= 69
\]

Answers:

renaming
associative property
commutative property
associative property
associative property
renaming
associative property
renaming

Exercises 9

A. Given: 57 + 22 = 79 because:

\[
57 + 22 = (50 + 7) + (20 + 2)
= 50 + (7 + (20 + 2))
= 50 + ((20 + 2) + 7)
= 50 + (20 + (2 + 7))
= (50 + 20) + (2 + 7)
= 70 + 9
= 79
\]

B. Given: 27 + 42 = 69 because

\[
27 + 42 = (20 + 7) + (40 + 2)
= 20 + (7 + (40 + 2))
= 20 + ((40 + 2) + 7)
= 20 + (40 + (2 + 7))
= (20 + 40) + (2 + 7)
= 60 + 9
= 69
\]

Answers:

renaming
associative property
commutative property
associative property
associative property
renaming
associative property

C. Given: \(38 + 25 = 63\) because

\[
38 + 25 = (30 + 8) + (20 + 5) \\
= 30 + (8 + (20 + 5)) \\
= 30 + ((20 + 5) + 8) \\
= 30 + (20 + (5 + 8)) \\
= (30 + 20) + (5 + 8) \\
= 50 + (10 + 3) \\
= (50 + 10) + 3 \\
= 60 + 3 \\
= 63
\]

D. Given: \(57 + 46 = 103\) because

\[
57 + 46 = (50 + 7) + (40 + 6) \\
= 50 + (7 + (40 + 6)) \\
= 50 + ((40 + 6) + 7) \\
= 50 + (40 + (6 + 7)) \\
= (50 + 40) + (6 + 7) \\
= 90 + (10 + 3) \\
= 100 + 3 \\
= 103
\]
SECTION 4. **Subtraction Algorithms**

**TASK 10**

Given two numbers $mn$ and $pq$, where each digit of $pq$ is less than or equal to the corresponding digit of $mn$, justify the subtraction algorithm by constructing a concrete representation.

**RULE 10**

Represent $mn$ and $pq$ as in Rule 8. Delete $q$ of the $n$ tally marks, leaving $s$ and $p$ of the $m$ groups of ten, leaving $r$. The difference is $rs$.

**EXAMPLES 10**

A. Given: 35 and 21

Answer:

35 can be represented

21 can be represented

Deleting 1 tally mark and 2 groups of ten from the representation of 35 gives

Hence, $35 - 21 = 14$.

B. Given: 27 and 23

Answer:

27 can be represented

23 can be represented

Deleting 3 tally marks and 2 groups of ten from the representation of 27 gives

Hence, $27 - 23 = 4$. 

201
Hence, 27 - 23 = 4.

EXERCISES 10

A. 34 and 22
B. 73 and 33
C. 29 and 24
D. 73 and 61

TASK 11

Given a subtraction problem, \( mn - pq \), which requires borrowing, represent the "borrowing" procedure algebraically.

RULE 11

Write

\[
mn = \text{m tens + n units} = (m - 1) \text{ tens} + (10 + n) \text{ units} = (m-1)(1n)
\]

\[
- pq = -(p \text{ tens + q units}) = -(p \text{ tens + q units}) = -pq
\]

Now subtract q from (10 + n) and p from (m - 1).

EXAMPLES 11

A. Given: \( 43 - 27 \)

Answer:

\[
43 = 4 \text{ tens + 3 units} = 3 \text{ tens + 13 units} = \begin{array}{c} 3 \text{ 1} \\
\end{array}
\]

\[
-27 = -(2 \text{ tens + 7 units}) = -(2 \text{ tens + 7 units}) = -27
\]

B. Given: \( 54 - 39 \)

Answer:

\[
54 = 5 \text{ tens + 4 units} = 4 \text{ tens + 14 units} = \begin{array}{c} 4 \text{ 1} \\
\end{array}
\]

\[
-39 = -(3 \text{ tens + 9 units}) = -(3 \text{ tens + 9 units}) = -39
\]
EXERCISES 11

A. 53 - 27
B. 45 - 39
C. 72 - 53
D. 65 - 38

TASK 12

Given a subtraction problem, \( mn - pq \), which requires borrowing, represent the "equal additions" procedure algebraically.

RULE 12

Increase \( n \) by 10 and \( p \) by 1 as indicated:

\[
\begin{align*}
\text{Increase } n \text{ by } 10 \text{ and } p \text{ by } 1 \text{ as indicated:} \\
mn &= m \text{ tens} + n \text{ units} \rightarrow m \text{ tens} + (10 + n) \text{ units} = m_{n+1} \\
-pq &= -(p \text{ tens} + q \text{ units}) \rightarrow -(p + 1) \text{ tens} + q \text{ units} = -(p+1)q \\
\end{align*}
\]

Now subtract \( q \) from \((10 + n)\) and \((p + 1)\) from \( m \):

\[
\begin{align*}
\text{Now subtract } q \text{ from } (10 + n) \text{ and } (p + 1) \text{ from } m &= (m - (p + 1))(10 - q)
\end{align*}
\]

EXAMPLES 12

A. Given: 45 - 18

Answer:

\[
\begin{align*}
45 &= 4 \text{ tens} + 5 \text{ units} \rightarrow 4 \text{ tens} + 15 \text{ units} = 45 \\
-18 &= -(1 \text{ ten} + 8 \text{ units}) \rightarrow -(2 \text{ tens} + 8 \text{ units}) = -18
\end{align*}
\]

B. Given: 52 - 39

Answer:

\[
\begin{align*}
52 &= 5 \text{ tens} + 2 \text{ units} \rightarrow 5 \text{ tens} + 12 \text{ units} = 52 \\
-39 &= -(3 \text{ tens} + 9 \text{ units}) \rightarrow -(4 \text{ tens} + 9 \text{ units}) = -39
\end{align*}
\]
EXERCISES 12

A. 47 - 29  
B. 74 - 26  
C. 65 - 58  
D. 28 - 19

TASK 13

Given a subtraction problem, \( mn - pq \), which \textbf{requires} borrowing, represent the "complement method" procedure algebraically.

RULE 13

Find a number, say \( rt \), which added to \( pq \) gives 100 (i.e., find the "complement" of \( pq \)). Add \( rt \) to \( pq \) and to \( mn \). Then subtract the new numbers as indicated.

\[
\begin{align*}
mn & \rightarrow mn + rt = 100 \\
- pq & \rightarrow -(pq + rt) = -100
\end{align*}
\]

EXAMPLES 13

A. Given: 54 - 37

Answer:

\[
\begin{align*}
54 & \rightarrow 54 + 63 = 117 \\
-37 & \rightarrow -(37 + 63) = -100
\end{align*}
\]

\[17\]

B. Given: 75 - 28

Answer:

\[
\begin{align*}
75 & \rightarrow 75 + 72 = 147 \\
-28 & \rightarrow -(28 + 72) = -100
\end{align*}
\]

\[47\]
EXERCISES 13

A. 47 - 28  
B. 93 - 65  
C. 74 - 68  
D. 67 - 49

TASK 14

Given a subtraction problem, \( mn - pq \), which requires borrowing, justify "borrowing" by constructing a concrete representation.

RULE 14

Represent \( mn \) and \( pq \) as in Rule 8. Delete one of the \( m \) groups of ten, giving \( m-1 \) groups, and replace it by adding 10 tally marks to the \( n \) available. Delete \( q \) of these tally marks leaving, say, \( s \), and also delete \( p \) groups of ten from the \( m-1 \) remaining, leaving, say, \( r \). The difference is \( rs \).

EXAMPLES 14

A. Given: 35 - 17

Answer:

35 can be represented

\[ \begin{array}{c}
\text{\rotatebox{90}{\$}} \\
\text{\rotatebox{90}{\$}} \\
\text{\rotatebox{90}{\$}} \\
\text{\rotatebox{90}{$\cdots$}}
\end{array} \]

17 can be represented

\[ \begin{array}{c}
\text{\rotatebox{90}{$\cdots$}} \\
\text{\rotatebox{90}{$\cdots$}} \\
\text{\rotatebox{90}{$\cdots$}} \\
\text{\rotatebox{90}{$\cdots$}} \\
\text{\rotatebox{90}{$\cdots$}}
\end{array} \]

Delete one of the 3 groups of ten in 35 and replace it with 10 tally marks

35

\[ \begin{array}{c}
\text{\rotatebox{90}{\$}} \\
\text{\rotatebox{90}{\$}} \\
\text{\rotatebox{90}{\$}} \\
\text{\rotatebox{90}{$\cdots$}}
\end{array} \]

Deleting 17 (1 group of ten and 7 tally marks) from this latter representation gives

\[ \begin{array}{c}
\text{\rotatebox{90}{$\cdots$}} \\
\text{\rotatebox{90}{$\cdots$}} \\
\text{\rotatebox{90}{$\cdots$}} \\
\text{\rotatebox{90}{$\cdots$}}
\end{array} \]

Hence, \( 35 - 17 = 18 \).
B. Given: 63 - 35.

Answer:

63 can be represented

35 can be represented

Delete one of the 6 groups of ten and replace it with 10 tally marks

Deleting 35 (3 groups of ten and 5 tally marks) from this latter representation gives

Hence, 63 - 35 = 28.

EXERCISES 14

A. 52 - 35
B. 47 - 39
C. 65 - 28
D. 34 - 16

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TASK 15 (Optional)

Given an algebraic justification of the (a) "borrowing," (b) "equal additions," or (c) "complement" methods of subtraction, supply the reason for each step in the justification.

RULE 15

A. Rule 9C.
B. If an expression of the form \((k + l) - (m + n)\) is changed to \((k - m) + (l - n)\) (or vice-versa) then the reason for this step is a special property of the Natural Numbers. (See Rule 39, Chapter 5).
C. If 0 is added to an expression, then the reason for this step is the additive identity property.

EXAMPLES 15

A. Given: \(83 - 47 = 36\) because

\[
83 - 47 = (80 + 3) - (40 + 7) \\
= (70 + 13) - (40 + 7) \\
= (70 - 40) + (13 - 7) \\
= 30 + 6 \\
= 36
\]

Answers:
- renaming
- property of the Natural Numbers

B. Given: \(84 - 65 = 19\) because

\[
84 - 65 = (84 - 65) + 0 \\
= (84 - 65) + (10 - 10) \\
= (84 + 10) - (65 + 10) \\
= (80 + 14) - (70 + 5) \\
= (80 - 70) + (14 - 5) \\
= 10 + 9 \\
= 19
\]

Answers:
- additive identity
- renaming
- property of the Natural Numbers
- renaming
- property of the Natural Numbers
- renaming
- renaming

C. Given: \(83 - 57 = 26\) because

\[
83 - 57 = (83 - 57) + 0 \\
= (83 - 57) + (43 - 43) \\
= (83 + 43) - (57 + 43) \\
= 126 - 100 \\
= 26
\]

Answers:
- additive identity
- renaming
- property of the Natural Numbers
- renaming
- renaming

EXERCISES 15

A. \(94 - 28 = 66\) because

\[
94 - 28 = (90 + 4) - (20 + 8) \\
= (80 + 14) - (20 + 8) \\
= (80 - 20) + (14 - 8)
\]

Answers:
B. Given: 75 - 47 = 28 because

\[75 - 47 = (70 + 5) - (40 + 7)\]
\[= (60 + 15) - (40 + 7)\]
\[= (60 - 40) + (15 - 7)\]
\[= 20 + 8\]
\[= 28\]

C. Given: 53 - 34 = 19 because

\[53 - 34 = (53 - 34) + 0\]
\[= (53 - 34) + (10 - 10)\]
\[= (53 + 10) - (34 + 10)\]
\[= (50 + 13) - (40 + 4)\]
\[= (50 - 40) + (13 - 4)\]
\[= 10 + 9\]
\[= 19\]

D. Given: 95 - 69 = 26 because

\[95 - 69 = (95 - 69) + 0\]
\[= (95 - 69) + (10 - 10)\]
\[= (95 + 10) - (69 + 10)\]
\[= (90 + 15) - (70 + 9)\]
\[= (90 - 70) + (15 - 9)\]
\[= 20 + 6\]
\[= 26\]

E. Given: 63 - 47 = 16 because

\[63 - 47 = (63 - 47) + 0\]
\[= (63 - 47) + (53 - 53)\]
\[= (63 + 53) - (47 + 53)\]
\[= 116 - 100\]
\[= 16\]

F. Given: 54 - 15 = 39 because

\[54 - 15 = (54 - 15) + 0\]
\[= (54 - 15) + (85 - 85)\]
\[= (54 + 85) - (15 + 85)\]
\[= 139 - 100\]
\[= 39\]
SECTION 5. Multiplication Algorithm

TASK 16

Given a pair of numbers, \( mn \) and \( pg \), justify the multiplication algorithm by constructing a concrete representation involving cartesian products.

RULE 16

Construct a rectangular array having \( 10m + n \) rows and \( 10p + q \) columns. Draw a horizontal line at the \( n \)th row and a vertical line at the \( q \)th column. Label the regions thus formed as follows: Upper left region = \( A \); upper right region = \( B \); lower left region = \( C \); and lower right region = \( D \). Then, \( A \) represents \( n \times q \), \( B \) represents \( n \times 10p \) and \( A + B \) represents the first partial sum in the algorithm. \( C \) represents \( 10m \times q \) and \( D \) represents \( 10m \times 10p \). \( C + D \) represent the second partial sum in the algorithm. \( (A + B) + (C + D) \) represents the product (third row in the algorithm).

EXAMPLES 16

A. Given: \[
\begin{array}{c}
23 \\
x15 \\
115 \\
23 \\
345
\end{array}
\]
Answer:
\[
\begin{array}{c}
5 \\
10 \\
3
\end{array}
\]
\[
\begin{array}{c}
A \\
B \\
C \\
D
\end{array}
\]

A represents \( 5 \times 3 = 15 \), \( B \) represents \( 5 \times 20 = 100 \), \( A + B = 115 \); \( C \) represents \( 10 \times 3 = 30 \), \( D \) represents \( 10 \times 20 = 200 \), \( C + D = 230 \); \( (A + B) + (C + D) = 345 \).

B. Given: \[
\begin{array}{c}
57 \\
x32 \\
114 \\
171 \\
1824
\end{array}
\]
Answer:
\[
\begin{array}{c}
7 \\
30 \\
2
\end{array}
\]
\[
\begin{array}{c}
A \\
B \\
C \\
D
\end{array}
\]

A represents \( 2 \times 7 = 14 \), \( B \) represents \( 2 \times 50 = 100 \), \( A + B = 114 \), \( C \) represents \( 30 \times 7 = 210 \), \( D \) represents \( 30 \times 50 = 1500 \), \( C + D = 1710 \); \( (A + B) + (C + D) = 1824 \).
EXERCISES 16

A. \[
\begin{array}{c}
42 \\
\times 17 \\
294 \\
42 \\
714 \\
\end{array}
\]

B. \[
\begin{array}{c}
85 \\
\times 52 \\
425 \\
4420 \\
\end{array}
\]

C. \[
\begin{array}{c}
39 \\
\times 21 \\
78 \\
819 \\
\end{array}
\]

D. \[
\begin{array}{c}
62 \\
\times 27 \\
124 \\
1674 \\
\end{array}
\]
SECTION 6. Division Algorithm

TASK 17

Given two numbers \( d \) and \( n \), \( d < n \), justify the division algorithm by constructing a concrete representation.

RULE 17

1. Construct an array of \( n \) tally marks with \( d \) elements in each column (plus a final column on the right, if necessary, with \( r < d \) elements). The number of columns which contain exactly \( d \) elements corresponds to the quotient, \( q \). The number of elements in the final column, if any, corresponds to the remainder, \( r \).

2. Perform the algorithm.

3. To see the relationship between the array and the algorithm, start from the left and group every ten columns into one group of ten. Group every ten groups of ten columns into one group of one hundred. Group every ten groups of one hundred groups into one group of one thousand, and so on. Count the groups containing the largest power of ten (tally marks). This number corresponds to the first step -- the left most digit in the quotient (in the algorithm). Then count the group containing the second highest power of ten (tally marks). This number corresponds to the second step -- the second digit from the left in the quotient (in the algorithm). Continue this process until all such groups have been accounted for. The remainder, \( r \), in the algorithm corresponds, as before, to the number of tally marks, if any, in the final column and \( 0 < r < d \).

EXAMPLES 17

A. Given: \( 4 \div 23 \)

Answer:

\[
\begin{array}{c|c|c}
\text{q} & \text{n} = 23 \\
\hline
5 & 5 \\
\hline
\text{d=4} & \text{r=3} \\
\hline
\end{array}
\]

B. Given: \( 3 \div 72 \)

Answer:

\[
\begin{array}{c|c|c|c|c}
\text{q} & \text{n} = 72 & \text{r=0} \\
\hline
2 & 4 \\
\hline
\text{d=3} & \text{d=3} & \text{d=3} & \text{d=3} \\
\hline
\end{array}
\]

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EXERCISES 17

A. \( \frac{5}{13} \)
B. \( \frac{6}{21} \)
C. \( \frac{4}{58} \)
D. \( \frac{12}{96} \)
E. \( \frac{13}{175} \)

Task 18 (Optional)

Given an algebraic justification of the usual (a) division, or (b) multiplication algorithms, supply the reason for each step in the justification.

Rule 18

A. Rule 9A
B. Rule 9C
C. If an expression of the form \((J + m) \times n\) is changed to \((J \times n) + (m \times n)\) (or vice-versa), then the reason for this step is the **distributive property** of multiplication over addition.
D. If a number \(m\) is written as \((p \times q) + r\), where \(r\) is less than \(p\), then the reason for this step is the **Euclidean Algorithm**.
EXAMPLES 18

A. Given: $256 \div 13 = 19 \; r \; 9$ because

\begin{align*}
256 &= 200 + 56 \\
&= [(15 \times 13) + 5] + 56 \quad \text{Euclidean Algorithm} \\
&= (15 \times 13) \div (5 + 56) \quad \text{associative property} \\
&= (15 \times 13) + 61 \quad \text{renaming} \\
&= (15 \times 13) + [(4 \times 13) + 9] \quad \text{Euclidean Algorithm} \\
&= [(15 \times 13) + (4 \times 13)] + 9 \quad \text{associative property} \\
&= [(15 + 4) \times 13] + 9 \quad \text{distributive property} \\
&= (19 \times 13) + 9 \quad \text{renaming}
\end{align*}

B. Given: $56 \times 42 = 2352$ because

\begin{align*}
56 \times 42 &= 56 \times (40 + 2) \\
&= (56 \times 40) + (56 \times 2) \quad \text{distributive property} \\
&= [(50 + 6) \times 40] + [(50 + 6) \times 2] \quad \text{renaming} \\
&= [(50 \times 40) + (6 \times 40)] + [(50 \times 2) + (6 \times 2)] \quad \text{distributive property} \\
&= (2000 + 240) + (100 + 12) \quad \text{renaming} \\
&= 2240 + 112 \quad \text{renaming} \\
&= 2352 \quad \text{renaming}
\end{align*}

EXERCISES 18

A. $77 \div 3 = 25 \; r \; 2$ because

\begin{align*}
77 &= 70 + 7 \\
&= [(20 \times 3) + 10] + 7 \\
&= (20 \times 3) + (10 + 7) \\
&= (20 \times 3) + 17 \\
&= (20 \times 3) + [(5 \times 3) + 2] \\
&= [(20 \times 3) + (5 \times 3)] + 2 \\
&= [(20 + 5) \times 3] + 2 \\
&= (25 \times 3) + 2
\end{align*}

B. $65 \times 38 = 2470$ because

\begin{align*}
65 \times 38 &= 65 \times (30 + 8) \\
&= (65 \times 30) + (65 \times 8) \\
&= [(60 + 5) \times 30] + [(60 + 5) \times 8] \\
&= [(60 \times 30) + (5 \times 30)] + [(60 \times 8) + (5 \times 8)] \\
&= (1800 + 150) + (480 + 40) \\
&= 1950 + 520 \\
&= 2470
\end{align*}

C. $36 \times 55 = 1980$ because
36 \times 55 = 36 \times (50 + 5) \\
= (36 \times 50) + (36 \times 5) \\
= [30 + 6 \times 50] + [(30 + 6) \times 5] \\
= [30 \times 50] + (6 \times 50) + [(30 \times 5) + (6 \times 5)] \\
= (1500 + 300) + (150 + 30) \\
= 1800 + 180 \\
= 1980 \\

D. \quad 89 \div 12 = 7 \times 5 \text{ because} \\
89 = 80 + 9 \\
= [(6 \times 12) + 8] + 9 \\
= (6 \times 12) + (8 + 9) \\
= (6 \times 12) + 17 \\
= (6 \times 12) + [(1 \times 12) + 5] \\
= [(6 \times 12) + (1 \times 12)] + 5 \\
= [(6 + 1) \times 12] + 5 \\
= (7 \times 12) + 5
CHAPTER 7

THE SYSTEM OF POSITIVE RATIONALS

SECTION 1. Fractions.

TASK 1

Given a pair of sets, name its fraction property.

RULE 1

Count the number of elements in each set. Write \( \frac{a}{b} \), where \( a \) is the number of elements in the first set and \( b \) is the number of elements in the second set.

EXAMPLES 1

A. Given: \( \{3, 5\}, \{4, 7, 2\} \). Answer: "2 to 3" since there are 2 elements in the first set and 3 elements in the second.

B. Given: \( \{1, 2, 3, 4, 5, 6\}, \{a, b, c, d\} \). Answer: "6 to 4" since there are 6 elements in the first set and 4 elements in the second.

EXERCISES 1

A. \( \{2, 4, 5\}, \{2, 4, 5, 6\} \)
B. \( \{a, b, c\}, \{c, d\} \)
C. \( \{e, f\}, \{g, h\} \)
D. \( \{1, 2, 3, 4, 5\}, \{a, b, c, d, e, f, g\} \)
E. \( \{\square, \circ, \sqrt{\ }, \Delta, \#\}, \{\square, \Delta, \#\} \)

TASK 2

Given a pair of sets, represent its fraction property as an ordered pair of
natural numbers.

RULE 2

Count the number of elements in each set. Write \((a, b)\), where \(a\) is the number of elements in the first set and \(b\) is the number of elements in the second set.

EXAMPLES 2

A. Given: \({2, 3, 7}\), \({a, b, 4, 5}\). Answer: \((3, 4)\) since there are 3 elements in the first set and 4 elements in the second.

B. Given: \({a, b, c, d, e, f, g}\), \({a, b, c}\). Answer: \((7, 3)\) since there are 7 elements in the first set and 3 elements in the second.

EXERCISES 2

A. \({5, 7}\), \({5, 7, 9, 10}\)
B. \({7, x, a}\), \({b}\)
C. \({x}\), \({1, 2, 3}\)
D. \({a, b, c}\), \({a, b, c}\)
E. \({a, b, c, d, e, f, g}\), \({\Box, \checkmark}\)

TASK 3

Given an ordered pair representation of the fraction property of a pair of sets, name the numerator and the denominator.

RULE 3

The first number of the ordered pair is the numerator and the second number is the denominator.

EXAMPLES 3

A. Given: \((7, 2)\). Answer: The numerator is 7 and the denominator is 2.
B. Given: \((2, 5)\). Answer: The numerator is 2 and the denominator is 5.

EXERCISES 3

A. \((6, 2)\)
B. \((4, 4)\)
C. \((1, 5)\)
D. \((3, 6)\)
E. \((12, 12)\)
TASK 4

Given a pair of sets, represent its fraction property as a "ratio":

RULE 4

Count the number of elements in each set. Write \( \frac{a}{b} \) where \( a \) is the number of elements in the first set and \( b \) is the number of elements in the second set.

EXAMPLES 4

A. Given: \([a, x], [a, x, z, b]\). Answer: 
   \( \frac{2}{4} \) since there are 2 elements in the first set and 4 elements in the second.

B. Given: \([1, 2, 3], [a, b]\). Answer: 
   \( \frac{3}{2} \) since there are 3 elements in the first set and 2 elements in the second.

EXERCISES 4

A. \([x, y, z], [x, y, z, w, t]\)
B. \([4, 5, 6], [4]\)
C. \([a, b, c, d], [1, 2, 3, 4]\)
D. \([7, 8, b, c, e], [x, y]\)
E. \([x], [1, 2, 3, 4, 5, 6, 7]\)

TASK 5

Given a pair of sets, represent its fraction property as a "fraction":

RULE 5

Count the number of elements in each set. Write \( \frac{a}{b} \) where \( a \) is the number of elements in the first set and \( b \) is the number of elements in the second set.

EXAMPLES 5

A. Given: \([1], [a, b, c]\). Answer: 
   \( \frac{1}{3} \) since there is 1 element in the first set and 3 elements in the second.

B. Given: \([x, y, z, w, t], [1, 2, 3]\). Answer: 
   \( \frac{5}{3} \) since there are 5 elements in the first set and 3 elements in the second.

EXERCISES 5

A. \([3, 7], [2, 3, 5, 6, 7]\)
B. \([a, b, c, d], [1]\)

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Given two pairs of sets, determine whether or not they are fractionally equivalent.

**RULE 6**

If the first sets of the given pairs can be placed in 1-1 correspondence (Rule 9, Chapter 2) and the second sets of the given pairs can be placed in 1-1 correspondence, then the (given) pairs of sets are fractionally equivalent; otherwise they are not.

**EXAMPLES 6**

A. Given: \( \{a, b\}, \{e, f, g\} \) and \( \{1, 2\}, \{1, 2, 3\} \). Answer:
\[ \begin{align*}
& a \leftrightarrow 1, \ b \leftrightarrow 2. \ \text{Therefore the first sets are in 1-1 correspondence.} \\
& e \leftrightarrow 1, \ f \leftrightarrow 2, \ g \leftrightarrow 3. \ \text{Therefore the second sets are in 1-1 correspondence.} \\
\end{align*} \]
\( \{a, b\}, \{e, f, g\} \) is fractionally equivalent to \( \{1, 2\}, \{1, 2, 3\} \).

B. Given: \( \{1, 2\}, \{3, 4, 5\} \) and \( \{a, b\}, \{a, b, c, d\} \). Answer:
\[ \begin{align*}
& 1 \leftrightarrow a, \ 2 \leftrightarrow b. \ \text{Therefore the first sets are in 1-1 correspondence.} \\
& 3 \leftrightarrow a, \ 4 \leftrightarrow b, \ 5 \leftrightarrow c. \ \text{The second sets cannot be placed in 1-1 correspondence.} \\
\end{align*} \]
\( \{1, 2\}, \{3, 4, 5\} \) is not fractionally equivalent to \( \{a, b\}, \{a, b, c, d\} \).

**EXERCISES 6**

A. \( \{x, y, z\}, \{1, 2, 3, 4, 5\} \) and \( \{1, 2, a, b, c\} \)
B. \( \{\#, 0\}, \{a, b, c, d, e\} \) and \( \{w, p\}, \{w, p, q, r, s, t\} \)
C. \( \{\#, 0, 2, 4\}, \{1, 3\} \) and \( \{x, p, r, s\}, \{g\} \)
D. \( \{f\}, \{f, g, h\} \) and \( \{e\}, \{2, 4, 6\} \)
E. \( \{h\}, \{1, 2, 3\} \) and \( \{a, b\}, \{c, d, e, f, g, h\} \)

---

**TASK 7**

Given a fraction \( \frac{a}{b} \), write two elements in the equivalence class of pairs of sets defined by \( \frac{a}{b} \).

**RULE 7**

Write two pairs of sets, each having \( \frac{a}{b} \) elements in the first set and \( \frac{b}{b} \) elements in the second set.
EXAMPLES 7

A. Given: \( \frac{2}{3} \). Answer:

\( ([a, b], [a, b, c]) \) and \( ([1, 2], [x, y, z]) \).

B. Given: \( \frac{4}{3} \). Answer:

\( ([a, f, g, h], [1, 7, 9]) \) and \( ([1, 2, 3, 4], [1, 2, 3]) \).

EXERCISES 7

A. \( \frac{3}{4} \)  B. \( \frac{1}{5} \)  C. \( \frac{4}{2} \)  D. \( \frac{5}{5} \)  E. \( \frac{6}{3} \)

TASK 8

Given a fraction \( \frac{a}{b} \), construct an interpretation in which the elements of both sets are equal sized parts of a whole.

RULE 8

Partition a whole (e.g. circle, square, etc.) into \( b \) equal sized pieces. Shade \( a \) of them. (Additional pieces of the same size may be added if needed.)

EXAMPLES 8

A. Given: \( \frac{1}{6} \). Construct:

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
& & & & & \\
\hline
& & & & & \\
\hline
\end{array}
\]

\[
\left\{ \begin{array}{c}
\square, \\
\square, \\
\square, \\
\square, \\
\square, \\
\square, \\
\square, \\
\square
\end{array} \right\} \quad \left\{ \begin{array}{c}
\square, \\
\square, \\
\square, \\
\square, \\
\square, \\
\square, \\
\square, \\
\square
\end{array} \right\}
\]

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B. Given: \( \frac{5}{4} \). Construct:

\[
\left( \left\{ \begin{array}{c}
\text{\includegraphics{rectangle.png}}
\end{array} \right\}, \left\{ \begin{array}{c}
\text{\includegraphics{rectangle.png}}
\end{array} \right\} \right)
\]

EXERCISES 8

A. \( \frac{1}{4} \)  
B. \( \frac{3}{3} \)  
C. \( \frac{3}{2} \)  
D. \( \frac{3}{6} \)  
E. \( \frac{4}{3} \)

TASK 9

Given a fraction \( \frac{a}{b} \), construct an interpretation in which the pair of sets have elements in common.

RULE 9

Construct a pair of sets where the first set \( A \) has "\( a \)" elements, the second set \( B \) has "\( b \)" elements and the larger set contains all of the elements of the smaller set. (If \( a = b \) make the two sets the same.)

EXAMPLES 9

A. Given: \( \frac{2}{3} \). Answer:

Let \( A = \{x, y\} \), \( B = \{x, y, z\} \)

\([x, y], \{x, y, z\}\) is the interpretation.

B. Given: \( \frac{5}{2} \). Answer:

Let \( A = \{1, 2, 3, 4, 5\} \), \( B = \{1, 2\} \)

\([\{1, 2, 3, 4, 5\}, \{1, 2\}\) is the interpretation.

EXERCISES 9

A. \( \frac{2}{4} \)  
B. \( \frac{4}{3} \)  
C. \( \frac{1}{1} \)  
D. \( \frac{3}{5} \)  
E. \( \frac{3}{2} \)
TASK 10

Given a fraction \( \frac{a}{b} \), construct an interpretation in which the elements of each set are themselves sets.

RULE 10

Construct a pair of sets where the first set has \( a \) elements, each of which is a set, and the second set has \( b \) elements, each of which is a set.

EXAMPLES 10

A. Given: \( \frac{1}{2} \). Construct:

\( ([\{1, 2, 3\}], [[x, y, z], [a, b, c]]) \).

B. Given: \( \frac{4}{3} \). Construct:

\( ([[1, 2}, [a, b]}, [x, y], [7, 9]], [[7, 8}, [e, f]}, [x, y]]) \).

EXERCISES 10

A. \( \frac{2}{4} \)   B. \( \frac{3}{2} \)   C. \( \frac{1}{6} \)   D. \( \frac{2}{2} \)   E. \( \frac{5}{3} \)

---

TASK 11

Given an interpretation of a fraction, write the fraction.

RULE 11

Write \( \frac{a}{b} \) where "\( a \)" is the number of elements (these elements may be sets) of the first set and "\( b \)" is the number of elements of the second set of the given interpretation.

EXAMPLES 11

A. Given:

\( \left( \left\{ \begin{array}{c} \text{ } \end{array} \right\}, \left\{ \begin{array}{c} \text{ } \end{array} \right\} \right) \)
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Answer:
\[ \frac{1}{2} \text{, since there is 1 element in the first set and there are 2 elements in the} \]
second set of the given interpretation.

B. Given: \( \{[a, b, c], [1, 2, 3]\}, \{[e, f, g], [1, 7, 9], [4, x, z], [1, b, c]\} \).

Answer:
\[ \frac{1}{4} \text{, since there are 2 elements (actually sets) in the first set and 4 elements} \]
in the second set of the given interpretation.

EXERCISES 11
A. \( \{[1, 2], [1, 3]\}, \{[a, b], [c, d], [e, f]\} \)
B. \( \{[h, k, l, m, n, o, p], [z, t, w]\} \)
C. \( \{[\text{\#}, \text{\dagger}, \text{\$}], [\text{\$}, \text{\&}, \text{\#}, \text{\$}, \text{\&}, \text{\$}]\} \)
D. \( \{[2, 5, 6], [2, 5, 6, 7, 8, 9, 10, 11, 12]\} \)
E. \( \{[a, b, c, d], [1, 2, 3, 4]\}, \{[e, f, g, h]\} \)

---

TASK 12

Given a fraction \( \frac{a}{b} \), construct a "state-state" interpretation of it.

RULE 12

Construct a pair of sets with "a" elements in the first set and "b" elements in the
second set.

EXAMPLES 12
A. Given: \( \frac{5}{4} \). Construct:
\( \{[1, 2, 3, 4, 5], [2, 3, 4, 5]\} \).
B. Given: \( \frac{1}{4} \). Construct:
\( \{[\text{\&}], [\text{\&}, \text{\$}, \text{\$}, \text{\$}]\} \).

EXERCISES 12
A. \( \frac{2}{5} \)  
B. \( \frac{3}{2} \)  
C. \( \frac{1}{7} \)  
D. \( \frac{3}{3} \)  
E. \( \frac{9}{4} \)

---

TASK 13

Given a fraction \( \frac{a}{b} \), construct an "operator-operator" interpretation.
RULE 13

Describe a situation in which a "whole" is partitioned into \( \frac{b}{a} \) equal sized pieces and \( \frac{a}{b} \) of them are chosen.

EXAMPLES 13

A. Given: \( \frac{2}{4} \). Answer:

A brick of ice cream is cut into 4 equal parts and one piece is given to Mary, one to John and one to Bill. The remaining piece is put back in the freezer.

B. Given: \( \frac{8}{8} \). Answer:

Mrs. Jones cut Mary's birthday cake into 8 equal sized pieces and distributed one piece to each of the 8 children attending the party.

EXERCISES 13

A. \( \frac{1}{4} \)  B. \( \frac{3}{5} \)  C. \( \frac{2}{2} \)  D. \( \frac{4}{6} \)  E. \( \frac{3}{7} \)

_______  •  _______

TASK 14

Given a fraction \( \frac{a}{b} \), construct a "state-operator" interpretation where the denominator is viewed as a state and the numerator as an operator.

RULE 14

Describe a situation in which there is a set with \( \frac{b}{a} \) elements and \( \frac{a}{b} \) of them are selected.

EXAMPLES 14

A. Given: \( \frac{4}{7} \). Answer:

Wendy has seven mystery books. She chooses four of them to lend to her best friend Hope.

B. Given: \( \frac{2}{5} \). Answer:

Betsy has five puzzles. Her brother Steven has chicken pox so Betsy picks two of the puzzles to let Steven play with while he is ill.
EXERCISES 14

A. \( \frac{1}{3} \)  
B. \( \frac{5}{6} \)  
C. \( \frac{4}{5} \)  
D. \( \frac{2}{4} \)  
E. \( \frac{3}{6} \)

TASK 15

Given a fraction \( \frac{a}{b} \), construct a "state-operator" interpretation where the numerator is viewed as a state and the denominator as an operator.

RULE 15

Describe a situation in which there is a set with "a" elements and each element is divided into "b" equivalent parts.

EXAMPLES 15

A. Given \( \frac{3}{4} \). Answer:

Jack has 3 oranges and wants to share them with four of his friends. He cuts each orange into four equivalent parts and each of the boys receives 3 of the pieces.

B. Given \( \frac{5}{3} \). Answer:

Mrs. Herman has five tomatoes and wants to share them equally with her husband and son. She cuts each tomato into 3 equivalent parts and gives 5 of the pieces to her husband, her son and to herself.

EXERCISES 15

A. \( \frac{1}{3} \)  
B. \( \frac{2}{5} \)  
C. \( \frac{4}{3} \)  
D. \( \frac{5}{7} \)  
E. \( \frac{3}{2} \)
SECTION 2. Positive Rational Numbers

TASK 16

Given two fractions, determine whether or not the fractions are equivalent.

RULE 16

Construct interpretations of the fractions in which the elements are equal sized parts of the same whole by applying Rule 8. If the (total) quantities represented by each of the interpretations are equal, then the (stated) fractions are equivalent; otherwise they are not.

EXAMPLES 16

A. Given: \( \frac{1}{3} \) and \( \frac{2}{6} \). Answer:

An interpretation of \( \frac{1}{3} \) is

An interpretation of \( \frac{2}{6} \) is

The total quantities represented are equal therefore \( \frac{1}{3} \) and \( \frac{2}{6} \) are equivalent.

B. Given: \( \frac{1}{2} \) and \( \frac{3}{4} \). Answer:

An interpretation of \( \frac{1}{2} \) is

An interpretation of \( \frac{3}{4} \) is

The total quantities represented are not equal therefore \( \frac{1}{2} \) and \( \frac{3}{4} \) are not equivalent.
EXERCISES 16

A. 2/4 and 1/2  
B. 1/6 and 2/3  
C. 3/4 and 6/8  
D. 2/8 and 1/4  
E. 3/12 and 2/6

TASK 17

Given two fractions, determine whether or not the fractions are equivalent.

RULE 17

Multiply the numerator of one fraction by the denominator of the other and vice versa. If the two products are the same, then the fractions are equivalent; otherwise they are not. (This rule uses the algebraic definition of equivalence (i.e., \( \frac{a}{b} \equiv \frac{c}{d} \) if and only if \( ad = bc \).)

EXAMPLES 17

A. Given \( \frac{2}{3} \) and \( \frac{8}{12} \). Answer:

\[ 2 \times 12 = 24 \text{ and } 3 \times 8 = 24. \text{ Therefore } \frac{2}{3} \text{ and } \frac{8}{12} \text{ are equivalent.} \]

B. Given \( \frac{4}{5} \) and \( \frac{11}{15} \). Answer:

\[ 4 \times 15 = 60 \text{ and } 5 \times 11 = 55. \text{ Therefore } \frac{4}{5} \text{ and } \frac{11}{15} \text{ are not equivalent.} \]

EXERCISES 17

A. \( \frac{3}{4} \) and \( \frac{7}{12} \)  
B. \( \frac{5}{2} \) and \( \frac{10}{4} \)  
C. \( \frac{4}{7} \) and \( \frac{20}{35} \)  
D. \( \frac{3}{8} \) and \( \frac{5}{12} \)  
E. \( \frac{5}{16} \) and \( \frac{22}{64} \)
SECTION 3. Representing Positive Rational Numbers.

3.1. Fractional Representation of Positive Rationals

TASK 18

Given two fractions \( \frac{a}{b} \) and \( \frac{c}{d} \), determine whether or not they represent the same rational number.

RULE 18

Apply Rule 17. If the two fractions are equivalent, then they represent the same rational number; otherwise they do not.

EXAMPLES 18

A. Given: \( \frac{4}{7} \) and \( \frac{12}{21} \). Answer:

Applying Rule 17 see that \( 7 \times 12 = 4 \times 21 \). So \( \frac{4}{7} \) and \( \frac{12}{21} \) are equivalent and, therefore, represent the same rational number.

B. Given: \( \frac{2}{3} \) and \( \frac{5}{9} \). Answer:

Applying Rule 17 see that \( 3 \times 5 \neq 2 \times 9 \). So \( \frac{2}{3} \) and \( \frac{5}{9} \) are not equivalent, and therefore, do not represent the same rational number.

EXERCISES 18

A. \( \frac{4}{3} \) and \( \frac{20}{15} \)
B. \( \frac{7}{9} \) and \( \frac{4}{7} \)
C. \( \frac{15}{32} \) and \( \frac{75}{160} \)
D. \( \frac{2}{11} \) and \( \frac{24}{132} \)

---

TASK 19

Given a fraction \( \frac{a}{b} \), write the rational number represented by \( \frac{a}{b} \).

RULE 19

Reduce \( \frac{a}{b} \) to lowest terms by expressing the numerator and denominator as a product of primes, and eliminating all primes occurring in both. Let \( \frac{c}{d} \) be the fraction reduced to the lowest terms. Then form the set

\[ \{ \frac{c}{d}, \frac{2c}{2d}, \frac{3c}{3d}, \frac{4c}{4d}, \ldots \} \].

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EXAMPLES 19

A. Given: \( \frac{8}{14} \). Answer:

\[
\frac{8}{14} = \frac{8 \times 2 \times 2}{7 \times 2 \times 7} = \frac{4}{7},
\]

\( \{ \frac{4}{7}, \frac{8}{14}, \frac{12}{21}, \frac{16}{28}, \ldots \} \) is the required rational number.

B. Given: \( \frac{84}{18} \). Answer:

\[
\frac{84}{18} = \frac{4 \times 21}{2 \times 9} = \frac{4 \times 2 \times 7}{2 \times 3 \times 3} = \frac{14}{3},
\]

\( \{ \frac{14}{3}, \frac{28}{6}, \frac{42}{9}, \frac{56}{12}, \ldots \} \) is the required rational number.

EXERCISES 19

A. \( \frac{2}{3} \)  
B. \( \frac{8}{12} \)  
C. \( \frac{7}{3} \)  
D. \( \frac{14}{42} \)  
E. \( \frac{18}{8} \)

TASK 20

Given a rational number, express it in canonical form.

RULE 20

Choose any fraction in the given set. Reduce it to lowest terms (as in Rule 19). Write the reduced fraction. (This is the canonical representation.)

EXAMPLES 20

A. Given: \( \{ \frac{4}{6}, \frac{2}{3}, \frac{6}{9}, \frac{8}{12}, \ldots, \frac{22}{33}, \frac{24}{36}, \ldots \} \). Answer:

Choose \( \frac{24}{36} \).

\[
\frac{24}{36} = \frac{4 \times 6}{6 \times 6} = \frac{4 \times 2}{6 \times 3} = \frac{2}{3},
\]

\( \frac{2}{3} \) is the canonical representative.
B. Given: \[ \frac{7}{2}, \frac{14}{4}, \frac{21}{6}, \frac{28}{8}, \ldots, \frac{70}{20}, \frac{77}{22}, \frac{84}{24}, \ldots \]. Answer:

Choose \( \frac{28}{8} \).

\[ \frac{28}{8} = \frac{8 \times 7}{8 \times 2} = \frac{7}{2}. \]

\( \frac{7}{2} \) is the canonical representative.

**EXERCISES 20**

A. \( \left\{ \frac{3}{12}, \frac{4}{16}, \frac{1}{4}, \frac{2}{8}, \frac{5}{20}, \frac{6}{24}, \ldots, \frac{9}{36}, \frac{10}{40}, \frac{11}{44}, \ldots \right\} \)

B. \( \left\{ \frac{7}{3}, \frac{14}{6}, \frac{21}{9}, \ldots, \frac{56}{24}, \frac{63}{27}, \frac{70}{30}, \ldots \right\} \)

C. \( \left\{ \frac{2}{5}, \frac{4}{10}, \frac{6}{15}, \ldots, \frac{14}{35}, \frac{16}{40}, \frac{18}{45}, \ldots \right\} \)

D. \( \left\{ \frac{2}{7}, \frac{4}{14}, \frac{6}{21}, \ldots, \frac{16}{56}, \frac{18}{63}, \frac{20}{70}, \ldots \right\} \)

E. \( \left\{ \frac{10}{8}, \frac{5}{12}, \frac{15}{16}, \ldots, \frac{35}{28}, \frac{40}{32}, \frac{45}{36}, \ldots \right\} \)

**TASK 21**

Given a set of fractions, find equivalent fractions all having the same denominator.

**RULE 21**

Find the least common multiple (L.C.M.) of the denominators (Rule 48, Chapter 5).

For each of the given fractions, find

\[ \frac{n \times \text{L.C.M.}}{\text{denominator}} \]

**EXAMPLES 21**

A. Given: \( \frac{1}{6}, \frac{2}{5} \) and \( \frac{4}{15} \). Answer:

\( 6 = 2 \times 3, 5 = 5 \times 1, 15 = 3 \times 5 \).

The L.C.M. of the denominators is \( 2 \times 3 \times 5 = 30 \) (Rule 48, Chapter 5).

For \( \frac{1}{6} \), \( n = \frac{1 \times 30}{6} = 5 \). \( \frac{1}{6} \) can be written as \( \frac{5}{30} \).
For \( \frac{2}{5} \), \( n = \frac{2 \times 30}{5} = 12 \). \( \frac{2}{5} \) can be written as \( \frac{12}{30} \).

For \( \frac{4}{15} \), \( n = \frac{4 \times 30}{15} = 8 \). \( \frac{4}{15} \) can be written as \( \frac{8}{30} \).

Therefore \( \frac{1}{6} \), \( \frac{2}{5} \) and \( \frac{4}{15} \) can be written as \( \frac{5}{30} \), \( \frac{12}{30} \) and \( \frac{8}{30} \).

B. Given \( \frac{2}{5} \), \( \frac{3}{4} \) and \( \frac{1}{10} \). Answer:

\( 5 = 5 \times 1 \), \( 4 = 2 \times 2 \), \( 10 = 2 \times 5 \)

The L.C.M. of the denominators is \( 2 \times 2 \times 5 = 20 \). (Rule 48, Chapter 5).

For \( \frac{2}{5} \), \( n = \frac{2 \times 20}{5} = 8 \). \( \frac{2}{5} \) can be written as \( \frac{8}{20} \).

For \( \frac{3}{4} \), \( n = \frac{3 \times 20}{4} = 15 \). \( \frac{3}{4} \) can be written as \( \frac{15}{20} \).

For \( \frac{1}{10} \), \( n = \frac{1 \times 20}{10} = 2 \). \( \frac{1}{10} \) can be written as \( \frac{2}{20} \).

Therefore \( \frac{2}{5} \), \( \frac{3}{4} \) and \( \frac{1}{10} \) can be written as \( \frac{8}{20} \), \( \frac{15}{20} \) and \( \frac{2}{20} \).

EXERCISES 21

A. \( \frac{1}{3} \), \( \frac{3}{4} \) and \( \frac{2}{5} \)

B. \( \frac{1}{4} \), \( \frac{3}{7} \) and \( \frac{5}{14} \)

C. \( \frac{3}{11} \) and \( \frac{5}{9} \)

D. \( \frac{2}{5} \), \( \frac{3}{15} \) and \( \frac{7}{20} \)

E. \( \frac{1}{6} \), \( \frac{3}{14} \) and \( \frac{2}{21} \)

TASK 22

Given a set of fractions, order the rational numbers they represent from largest to smallest.
RULE 22

Apply Rule 21 so that all fractions are expressed as fractions with the same denominator. Write the fraction with largest numerator first, second largest numerator second, etc.

EXAMPLES 22

A. Given $\frac{2}{3}$ and $\frac{14}{19}$. Answer:

The L.C.M. of 3 and 19 is 57.

$\frac{2}{3}$ can be written as $\frac{38}{57}$.

$\frac{14}{19}$ can be written as $\frac{42}{57}$.

$\frac{14}{19} > \frac{2}{3}$.

B. Given $\frac{3}{4}$, $\frac{2}{3}$ and $\frac{4}{5}$. Answer:

The L.C.M. of 4, 3 and 5 is 60.

$\frac{3}{4}$ can be written as $\frac{45}{60}$.

$\frac{2}{3}$ can be written as $\frac{40}{60}$.

$\frac{4}{5}$ can be written as $\frac{48}{60}$.

$\frac{4}{5} > \frac{3}{4} > \frac{2}{3}$.

EXERCISES 22

A. $\frac{1}{3}$, $\frac{3}{7}$ and $\frac{7}{18}$

B. $\frac{4}{7}$, $\frac{5}{8}$ and $\frac{9}{16}$

C. $\frac{7}{16}$ and $\frac{9}{19}$

D. $\frac{3}{5}$, $\frac{11}{20}$, $\frac{7}{12}$ and $\frac{8}{15}$

E. $\frac{15}{32}$ and $\frac{11}{24}$
SECTION 3.2. Decimal Representation of Positive Rationals

TASK 23

Given a terminating decimal, write it as a fraction.

RULE 23

Count the number of digits appearing to the right of the decimal point (say \( n \)). Write \( \frac{a}{10^n} \) where \( a \) represents the given number with the decimal point omitted and \( 10^n \) represents \( 10 \) multiplied by itself \( n \)-times.

EXAMPLES 23

A. Given: 3.216. Answer: Three digits appear to the right of the decimal point. \( 10^3 = 10 \times 10 \times 10 = 1000 \). 
3216 is the required fraction.
1000

B. Given: .78917. Answer: Five digits appear to the right of the decimal point. 
\( 10^5 = 10 \times 10 \times 10 \times 10 \times 10 = 100,000 \). 
78917 is the required fraction.
100,000

EXERCISES 23

A. 312.
B. 4.017
C. 3.12410
D. 13.4
E. .07306

TASK 24

Given a fraction \( \frac{a}{b} \), write its decimal representation.

RULE 24

Perform a "long" division of \( a \) by \( b \) after placing one decimal point to the right of the units digit of \( a \) and another directly above the first decimal point in the place reserved for the quotient. Add zeroes to \( a \) as necessary and continue the division until (i) a remainder of zero is obtained or (ii) a sequence of numbers keeps repeating in the quotient. In case (i), write the quotient. In case (ii) write the quotient and indicate the repetition by adjoining "..." after the repeating numbers.
EXAMPLES 24

A. Given: \( \frac{107}{50} \). Answer:

\[
\begin{array}{c|c}
& 2.14 \\
50 & 107.00 \\
\hline
100 & 70 \\
50 & 200 \\
200 & \\
\end{array}
\]

2.14 is the required representation.

B. Given: \( \frac{107}{33} \). Answer:

\[
\begin{array}{c|c}
& 3.2424 \\
33 & 107.0000 \\
\hline
99 & 80 \\
66 & 140 \\
132 & 8 \\
\end{array}
\]

3.2424... is the required representation.

EXERCISES 24

A. \( \frac{6}{25} \)

B. \( \frac{4}{33} \)

C. \( \frac{7}{9} \)

D. \( \frac{61}{20} \)

E. \( \frac{151}{500} \)

TASK 25

Given a fraction, determine whether or not it corresponds to a terminating decimal.
RULE 25

Reduce the fraction to lowest terms. Write the denominator as a product of primes. If the only prime factors are 2 and/or 5, then the (stated) fraction corresponds to a terminating decimal; otherwise it does not.

EXAMPLES 25

A. Given: \( \frac{7}{40} \). Answer:

\[ 40 = 8 \times 5 = 2 \times 2 \times 2 \times 5. \] The only prime factors are 2 and 5, therefore \( \frac{7}{40} \) corresponds to a terminating decimal.

B. Given: \( \frac{115}{132} \) Answer:

\[ 132 = 2 \times 66 = 2 \times 2 \times 33 = 2 \times 2 \times 3 \times 11. \] There are prime factors other than 2 and 5, therefore \( \frac{115}{132} \) does not correspond to a terminating decimal.

EXERCISES 25

A. \( \frac{7}{15} \)
B. \( \frac{43}{40} \)
C. \( \frac{87}{125} \)
D. \( \frac{83}{116} \)
E. \( \frac{69}{60} \)

TASK 26

Given a fraction, represent it as a non-terminating repeating decimal.

RULE 26

Divide numerator by denominator (as in Rule 24). If the decimal terminates, adjoin 0’s to quotient and "..." to the right of the zeroes. If the decimal does not terminate express repeating numbers as in Rule 24.

EXAMPLES 26

A. Given: \( \frac{14}{99} \). Answer:
\[
\begin{array}{c}
\underline{.1414} \\
99 / 14,0000 \\
\underline{9 9} \\
\underline{9 9} \\
\underline{4 10} \\
\underline{3 96} \\
\underline{140} \\
\underline{9 9} \\
\underline{4 10} \\
\underline{3 96} \\
\underline{14}
\end{array}
\]

.141414... is the required representation.

B. Given: \( \frac{391}{125} \)  

Answer:

\[
\begin{array}{c}
\underline{3.128} \\
125 / 391.000 \\
\underline{375} \\
16 0 \\
\underline{12 5} \\
3 50 \\
\underline{2 50} \\
1 000 \\
\underline{1 000}
\end{array}
\]

3.128000... is the required representation.

EXERCISES 26

A. \( \frac{3}{20} \)

B. \( \frac{17}{11} \)

C. \( \frac{71}{60} \)

D. \( \frac{8}{18} \)

E. \( \frac{8}{25} \)
SECTION 4. Addition

TASK 27

Given two rational numbers \( r_1 \) and \( r_2 \), find the sum \( r_1 + r_2 \) by using the definition of addition of rational numbers.

RULE 27 (Definition of addition of rational numbers)

Choose a fractional representative \( \frac{a}{b} \) of \( r_1 \) (i.e., any fraction from the set \( r_1 \)) and a fractional representative \( \frac{c}{d} \) of \( r_2 \) (i.e., any fraction from the set \( r_2 \)). Find \( \frac{(a \times d) + (b \times c)}{b \times d} \) and then find the rational number corresponding to this fraction (as in Rule 19).

EXAMPLES 27

A. Given: \( \left\{ \frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \ldots \right\} \) and \( \left\{ \frac{1}{4}, \frac{3}{8}, \frac{5}{16}, \ldots \right\} \). Answer:

Choose \( \frac{3}{6} \) and \( \frac{2}{8} \).

\[
\frac{(3 \times 8) + (6 \times 2)}{6 \times 8} = \frac{24 + 12}{48} = \frac{36}{48} = \frac{3}{4} \quad \text{and} \quad \frac{8 \times 3}{8 \times 4} = \frac{3}{4}.
\]

The sum is \( \left\{ \frac{3}{4}, \frac{5}{8}, \frac{7}{16}, \ldots \right\} \).

B. Given: \( \left\{ \frac{5}{3}, \frac{10}{6}, \frac{15}{9}, \frac{20}{12}, \ldots \right\} \) and \( \left\{ \frac{1}{6}, \frac{2}{12}, \frac{3}{18}, \frac{4}{24}, \ldots \right\} \). Answer:

Choose \( \frac{5}{3} \) and \( \frac{8}{48} \).

\[
\frac{(5 \times 48) + (3 \times 8)}{3 \times 48} = \frac{240 + 24}{144} = \frac{264}{144} = \frac{4 \times 3 \times 22}{4 \times 3 \times 4 \times 3} = \frac{2 \times 11}{2 \times 2 \times 3} = \frac{11}{6}
\]

The sum is \( \left\{ \frac{11}{6}, \frac{11}{12}, \frac{11}{18}, \frac{11}{24}, \ldots \right\} \).

EXERCISES 27

A. \( \left\{ \frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \ldots \right\} \) and \( \left\{ \frac{1}{5}, \frac{2}{10}, \frac{3}{15}, \frac{4}{20}, \ldots \right\} \)

B. \( \left\{ \frac{3}{4}, \frac{6}{8}, \frac{9}{12}, \frac{12}{16}, \ldots \right\} \) and \( \left\{ \frac{5}{4}, \frac{10}{8}, \frac{15}{12}, \frac{20}{16}, \ldots \right\} \)

C. \( \left\{ \frac{7}{6}, \frac{14}{12}, \frac{21}{18}, \frac{28}{24}, \ldots \right\} \) and \( \left\{ \frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots \right\} \)

D. \( \left\{ \frac{4}{3}, \frac{8}{6}, \frac{12}{9}, \frac{16}{12}, \ldots \right\} \) and \( \left\{ \frac{9}{8}, \frac{18}{16}, \frac{27}{24}, \frac{36}{32}, \ldots \right\} \)

E. \( \left\{ \frac{2}{7}, \frac{4}{14}, \frac{6}{21}, \frac{8}{28}, \ldots \right\} \) and \( \left\{ \frac{4}{7}, \frac{8}{14}, \frac{12}{21}, \frac{16}{28}, \ldots \right\} \)

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Given two rational numbers, $r_1$ and $r_2$, determine whether or not the closure law for addition of rational numbers holds for $r_1$ and $r_2$.

**RULE 28**

Apply Rule 19, Chapter 5 to Rule 11, Chapter 5 and then apply the derived rule to $r_1$ and $r_2$.

**EXERCISES 28**

A. \( \left\{ \frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \frac{4}{16}, \ldots \right\} \) and \( \left\{ \frac{4}{3}, \frac{8}{6}, \frac{12}{9}, \frac{16}{12}, \ldots \right\} \)

B. \( \left\{ \frac{2}{7}, \frac{4}{14}, \frac{6}{21}, \frac{8}{28}, \ldots \right\} \) and \( \left\{ \frac{3}{5}, \frac{6}{10}, \frac{9}{15}, \frac{12}{20}, \ldots \right\} \)

C. \( \left\{ \frac{1}{8}, \frac{2}{16}, \frac{3}{24}, \frac{4}{32}, \ldots \right\} \) and \( \left\{ \frac{3}{4}, \frac{6}{8}, \frac{9}{12}, \frac{12}{16}, \ldots \right\} \)

D. \( \left\{ \frac{2}{2}, \frac{6}{6}, \frac{9}{9}, \frac{12}{12}, \ldots \right\} \) and \( \left\{ \frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots \right\} \)

E. \( \left\{ \frac{1}{5}, \frac{2}{10}, \frac{3}{15}, \frac{4}{20}, \ldots \right\} \) and \( \left\{ \frac{4}{5}, \frac{8}{10}, \frac{12}{15}, \frac{16}{20}, \ldots \right\} \)

---

Given two rational numbers $r_1$ and $r_2$, show that the operation of addition of rational numbers is well-defined for $r_1$ and $r_2$.

**RULE 29**

Apply Rule 23, Chapter 5 to $r_1$ and $r_2$.

**EXERCISES 29**

A. \( \left\{ \frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \ldots \right\} \) and \( \left\{ \frac{2}{8}, \frac{18}{16}, \frac{27}{24}, \frac{36}{32}, \ldots \right\} \)

B. \( \left\{ \frac{3}{4}, \frac{6}{8}, \frac{9}{12}, \frac{12}{16}, \ldots \right\} \) and \( \left\{ \frac{4}{7}, \frac{8}{14}, \frac{12}{21}, \frac{16}{28}, \ldots \right\} \)

C. \( \left\{ \frac{7}{6}, \frac{14}{12}, \frac{21}{18}, \frac{28}{24}, \ldots \right\} \) and \( \left\{ \frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \ldots \right\} \)

D. \( \left\{ \frac{4}{3}, \frac{8}{6}, \frac{12}{9}, \frac{16}{12}, \ldots \right\} \) and \( \left\{ \frac{15}{4}, \frac{30}{8}, \frac{45}{12}, \frac{60}{16}, \ldots \right\} \)

E. \( \left\{ \frac{2}{7}, \frac{4}{14}, \frac{6}{21}, \frac{8}{28}, \ldots \right\} \) and \( \left\{ \frac{11}{4}, \frac{22}{8}, \frac{33}{12}, \frac{44}{16}, \ldots \right\} \)

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Given fractional representatives $\frac{a}{b}$ and $\frac{c}{d}$ of two different rational numbers, show that the fraction representing the amount obtained by combining interpretations of the given fractions (i.e., combining the amounts) is the same as the fraction obtained by applying the definition of addition to the given fractions themselves.

**Rule 30**

1. Find the sum by applying the definition of addition (as in Rule 27).
2. Construct an interpretation of $\frac{a}{b}$ and $\frac{c}{d}$ as equal sized parts of the same whole (as in Rule 8). Partition each of the $\frac{a}{b}$ pieces of size $\frac{1}{b}$ into $d$ equal sized parts and each of the $\frac{c}{d}$ pieces of size $\frac{1}{d}$ into $b$ equal sized parts. Combine these two amounts and write the sum as $\frac{t}{bd}$ where $t$ is the total number of equal parts and $bd$ is the size of each part.

**Examples 30**

A. Given $\frac{1}{3}$ and $\frac{1}{4}$. Answer:

1. Apply the definition (Rule 27) to obtain $\frac{1}{3} + \frac{1}{4} = \frac{(1 \times 4) + (3 \times 1)}{3 \times 4} = \frac{7}{12}$.
2. Partition $\frac{1}{3}$ into 4 equal sized pieces and $\frac{1}{4}$ into 3 equal sized pieces and combine to obtain 7 equal sized pieces each of size $\frac{1}{12}$ (i.e., $\frac{7}{12}$).

B. Given $\frac{1}{4}$ and $\frac{1}{6}$. Answer:

1. Apply the definition (Rule 27) to obtain $\frac{1}{4} + \frac{1}{6} = \frac{(1 \times 6) + (4 \times 1)}{4 \times 6} = \frac{10}{24}$.

(2) Partition $\frac{1}{4}$ into 6 equal sized pieces and $\frac{1}{6}$ into 4 equal sized pieces and combine to obtain
EXERCISES 30

A. \( \frac{1}{2} \) and \( \frac{1}{4} \)
B. \( \frac{1}{3} \) and \( \frac{1}{6} \)
C. \( \frac{3}{8} \) and \( \frac{1}{2} \)
D. \( \frac{1}{2} \) and \( \frac{1}{6} \)
E. \( \frac{3}{4} \) and \( \frac{1}{6} \)

TASK 31

Given two rational numbers \( x_1 \) and \( x_2 \) represented by fractions \( \frac{a}{b} \) and \( \frac{c}{d} \), find the sum using least common multiples.

RULE 31

Use Rule 21 to express each fraction as a fraction with denominator equal to the least common multiple of the denominators. The sum is represented by the sum of the obtained numerators over the least common multiple.

EXAMPLES 31

A. Given: \( \frac{2}{3} \) and \( \frac{3}{5} \). Answer:

The L.C.M. of 3 and 5 is 15 (Rule 48, Chapter 5).

\( \frac{2}{3} \) can be represented as \( \frac{10}{15} \) (Rule 21) and \( \frac{3}{5} \) can be represented as \( \frac{9}{15} \) (Rule 21).

\( \frac{2}{3} + \frac{3}{5} = \frac{10}{15} + \frac{9}{15} = \frac{19}{15} \).
Given \( \frac{1}{4} \) and \( \frac{3}{6} \). Answer:

The L.C.M. of 4 and 6 is 12 (Rule 48, Chapter 5).
\( \frac{1}{4} \) can be represented as \( \frac{3}{12} \) (Rule 21) and \( \frac{3}{6} \) can be represented as \( \frac{6}{12} \) (Rule 21).
\[ \frac{1}{4} + \frac{3}{6} = \frac{3}{12} + \frac{6}{12} = \frac{9}{12}. \]

EXERCISES 31

A. \( \frac{1}{3} \) and \( \frac{2}{4} \)
B. \( \frac{3}{5} \) and \( \frac{7}{10} \)
C. \( \frac{9}{8} \) and \( \frac{1}{12} \)
D. \( \frac{4}{6} \) and \( \frac{3}{14} \)
E. \( \frac{5}{12} \) and \( \frac{1}{16} \)

TASK 32

Given two rational numbers \( x_1 \) and \( x_2 \) represented by terminating decimals, find the sum (expressed as a decimal).

RULE 32

Write the numbers one below the other in such a way that the decimal points in the addends "line up." Add the numbers without regard to the decimal points and then place a decimal point in the sum below the other decimal points.

EXAMPLES 32

A. Given: 31.216 and 4.73. Answer:

\[
\begin{array}{c}
31.216 \\
+ 4.73 \\
\hline
35.946
\end{array}
\]

35.946 is the sum.

B. Given: .4372 and 15.01. Answer:

\[
\begin{array}{c}
.4372 \\
+15.01 \\
\hline
15.4472
\end{array}
\]

15.4472 is the sum.
EXERCISES 32

A. 42.36 and 1.7529  
B. .4072 and .7956  
C. 18.14 and .3298  
D. 7.0132 and 15.78  
E. .489 and 1.63

TASK 33

Given two rational numbers \( r_1 \) and \( r_2 \) represented by terminating decimals, justify the algorithm for addition, Rule 32, by expressing the given rationals as fractions with the same denominator, adding the fractions and changing the sum back to a decimal.

RULE 33

(1) Add the decimals (Rule 32). (2) Write each decimal as a fraction (Rule 23), then add the fractions (Rule 31), and finally express the sum as a decimal (Rule 24). (Notice that the results of (1) and (2) are identical.)

EXAMPLES 33

A. Given: .379 and .426. Answer:

\[
\begin{array}{c}
.379 \\
+ .426 \\
\hline
.805
\end{array}
\]

(Rule 32)

.379 can be written as \( \frac{379}{1000} \) and .426 can be written as \( \frac{426}{1000} \) (Rule 23).

\[
\frac{379}{1000} + \frac{426}{1000} = \frac{805}{1000}
\]

(Rule 31).

Apply Rule 24 to \( \frac{805}{1000} \) and obtain .805.

B. Given: 11.342 and 2.71. Answer:

\[
\begin{array}{c}
11.342 \\
+ 2.71 \\
\hline
14.052
\end{array}
\]

(Rule 32)

11.342 can be written as \( \frac{11342}{1000} \) and 2.71 can be written as \( \frac{271}{100} \) (Rule 23).

The L.C.M. of 100 and 1000 is 1000.

\[
\frac{11342}{1000} + \frac{271}{100} = \frac{11342}{1000} + \frac{2710}{1000} = \frac{14052}{1000}
\]

(Rule 31)

Apply Rule 24 to \( \frac{14052}{1000} \) and obtain 14.052.
EXERCISES 33

A. 0.219 and 1.357
B. 2.3 and 0.62
C. 4.1 and 1.937
D. 0.8724 and 1.013
E. 3.25 and 0.6
SECTION 5. Subtraction.

TASK 34

Given two rational numbers \( r_1 \) and \( r_2 \) (with \( r_1 > r_2 \)), use the definition of subtraction of rational numbers to find the difference \( r_1 - r_2 \).

RULE 34 (Definition of subtraction of rational numbers)

Choose fractional representatives \( \frac{a}{b} \) of \( r_1 \) and \( \frac{c}{d} \) of \( r_2 \). Find \( \frac{(a \times d) - (b \times c)}{b \times d} \) and then find the rational number corresponding to this fraction (as in Rule 19).

EXAMPLES 34

A. Given \( \{\frac{3}{4}, \frac{6}{8}, \frac{9}{12}, \frac{12}{16}, \ldots \} \) and \( \{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \ldots \} \). Answer:

Choose \( \frac{6}{8} \) and \( \frac{16}{32} \).

\[
\frac{(6 \times 32) - (8 \times 16)}{8 \times 32} = \frac{64 - 128}{256} = \frac{64 \times 1}{64 \times 4} = \frac{1}{4}
\]

The difference is \( \{\frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \frac{4}{16}, \ldots \} \).

B. Given \( \{\frac{7}{8}, \frac{14}{16}, \frac{21}{24}, \frac{28}{32}, \ldots \} \) and \( \{\frac{1}{6}, \frac{2}{12}, \frac{3}{18}, \frac{4}{24}, \ldots \} \). Answer:

Choose \( \frac{7}{8} \) and \( \frac{1}{6} \).

\[
\frac{(7 \times 6) - (8 \times 1)}{8 \times 6} = \frac{42 - 8}{48} = \frac{8 \times 17}{8 \times 24} = \frac{17}{24}
\]

The difference is \( \{\frac{17}{24}, \frac{34}{48}, \frac{51}{72}, \frac{68}{96}, \ldots \} \).

EXERCISES 34

A. \( \{\frac{7}{8}, \frac{14}{16}, \frac{21}{24}, \frac{28}{32}, \ldots \} \) and \( \{\frac{3}{4}, \frac{6}{8}, \frac{9}{12}, \frac{12}{16}, \ldots \} \)

B. \( \{\frac{5}{6}, \frac{10}{12}, \frac{15}{18}, \frac{20}{24}, \ldots \} \) and \( \{\frac{2}{5}, \frac{4}{10}, \frac{6}{15}, \frac{8}{20}, \ldots \} \)

C. \( \{\frac{1}{3}, \frac{2}{6}, \frac{3}{12}, \ldots \} \) and \( \{\frac{3}{10}, \frac{6}{20}, \frac{9}{30}, \frac{12}{40}, \ldots \} \)

D. \( \{\frac{3}{8}, \frac{6}{16}, \frac{9}{24}, \frac{12}{32}, \ldots \} \) and \( \{\frac{2}{7}, \frac{4}{14}, \frac{6}{21}, \frac{8}{28}, \ldots \} \)

E. \( \{\frac{9}{2}, \frac{18}{4}, \frac{27}{6}, \frac{36}{8}, \ldots \} \) and \( \{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots \} \)
Given an addition problem $a + b = c$, show that subtraction may be viewed as the inverse of addition by writing the corresponding subtraction problem.

**RULE 35**

Apply Rule 35, Chapter 5 to $a + b = c$.

**EXERCISES 35**

A. $a + \{ \frac{7}{6}, \frac{14}{16}, \frac{21}{24}, \frac{28}{32}, \ldots \} = \{ \frac{3}{2}, \frac{6}{4}, \frac{9}{6}, \frac{12}{8}, \ldots \}$

B. $a + \{ \frac{2}{5}, \frac{4}{10}, \frac{6}{15}, \frac{8}{20}, \ldots \} = \{ \frac{3}{4}, \frac{6}{8}, \frac{9}{12}, \frac{12}{16}, \ldots \}$

C. $a + \{ \frac{1}{2}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \ldots \} = \{ \frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \ldots \}$

D. $a + \{ \frac{16}{13}, \frac{32}{26}, \frac{48}{39}, \frac{64}{52}, \ldots \} = \{ \frac{12}{5}, \frac{24}{10}, \frac{36}{15}, \frac{48}{20}, \ldots \}$

E. $a + \{ \frac{7}{5}, \frac{14}{10}, \frac{21}{15}, \frac{28}{20}, \ldots \} = \{ \frac{13}{7}, \frac{26}{14}, \frac{39}{21}, \frac{52}{28}, \ldots \}$

---

Given two rational numbers $a_1$ and $a_2$ ($a_1 > a_2$), show that the operation of subtraction of rational numbers is well-defined for $a_1$ and $a_2$.

**RULE 36**

Apply Rule 23, Chapter 5 to $a_1$ and $a_2$.

**EXERCISES 36**

A. $a_1 + \{ \frac{1}{6}, \frac{2}{12}, \frac{3}{18}, \frac{4}{24}, \ldots \} and \{ \frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots \}$

B. $a_1 + \{ \frac{2}{5}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots \}$ and $\{ \frac{5}{4}, \frac{10}{8}, \frac{15}{12}, \frac{20}{16}, \ldots \}$

C. $a_1 + \{ \frac{5}{8}, \frac{10}{16}, \frac{15}{24}, \frac{20}{32}, \ldots \}$ and $\{ \frac{1}{5}, \frac{2}{10}, \frac{3}{15}, \frac{4}{20}, \ldots \}$

D. $a_1 + \{ \frac{10}{7}, \frac{20}{14}, \frac{30}{21}, \frac{40}{28}, \ldots \}$ and $\{ \frac{1}{7}, \frac{2}{14}, \frac{3}{21}, \frac{4}{28}, \ldots \}$

E. $a_1 + \{ \frac{1}{6}, \frac{2}{12}, \frac{3}{18}, \frac{4}{24}, \ldots \}$ and $\{ \frac{7}{3}, \frac{14}{6}, \frac{21}{9}, \frac{28}{12}, \ldots \}$

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Given two fractional representatives (of rational numbers) \( \frac{a}{b} \) and \( \frac{c}{d} \), show that the fraction representing the amount obtained by taking away an interpretation of the smaller fraction (an amount) from an interpretation of the larger fraction (a larger amount) is the same as the fraction obtained by applying the definition of subtraction to the given fractions themselves.

RULE 37

(1) Find the difference by applying the definition of subtraction (Rule 34).
(2) Construct an interpretation of \( \frac{a}{b} \) and \( \frac{c}{d} \) as equal sized parts of the same whole (Rule 8), where the smaller amount lies inside the larger amount. Partition each of the \( a \) pieces of size \( \frac{1}{b} \) into \( d \) equal sized parts and each of the \( c \) pieces of size \( \frac{1}{d} \) into \( b \) equal sized parts. Subtract the number of equal parts of the smaller from the number of equal parts of the larger and write this difference divided by \( bd \) (the size of each equal part).

EXAMPLES 37

A. Given \( \frac{2}{3} \) and \( \frac{1}{4} \). Answer:

\[
\text{Answer: } \frac{2}{3} - \frac{1}{4} = \frac{(2 \times 4) - (3 \times 1)}{3 \times 4} = \frac{5}{12}.
\]

Partition each of the 2 pieces of size \( \frac{1}{3} \) into 4 equal sized pieces and \( \frac{1}{4} \) into 3 equal sized pieces and subtract to obtain 5 pieces each of size \( \frac{1}{12} \) — i.e., \( \frac{5}{12} \).
B. Given $\frac{3}{4}$ and $\frac{2}{3}$. Answer:

(1) Apply the definition (Rule 34) to obtain $\frac{3}{4} - \frac{2}{3} = \frac{(3 \times 3) - (4 \times 2)}{4 \times 3} = \frac{1}{12}$.

Partition each of the 3 pieces of size $\frac{1}{4}$ into 3 equal sized pieces and each of the 2 pieces of size $\frac{1}{3}$ into 4 equal sized pieces and subtract to obtain

1 piece of size $\frac{1}{12}$ -- i.e., $\frac{1}{12}$.

EXERCISES 37

A. $\frac{1}{2}$ and $\frac{1}{6}$
B. $\frac{1}{3}$ and $\frac{1}{6}$
C. $\frac{1}{2}$ and $\frac{3}{8}$
D. $\frac{3}{4}$ and $\frac{1}{6}$
E. $\frac{1}{2}$ and $\frac{2}{6}$
TASK 38

Given two fractional representatives (of rational numbers), find the difference using least common multiples.

RULE 38

Apply Rule 4, Chapter 6 to Rule 31 (i.e., replace + by -) and apply the derived rule to \( r_1 \) and \( r_2 \).

EXERCISES 38

A. \( \frac{4}{5} \) and \( \frac{1}{3} \)
B. \( \frac{3}{7} \) and \( \frac{2}{9} \)
C. \( \frac{17}{12} \) and \( \frac{11}{9} \)
D. \( \frac{7}{8} \) and \( \frac{2}{3} \)
E. \( \frac{1}{4} \) and \( \frac{1}{5} \)

TASK 39

Given two fractional representatives (of rational numbers) having the same denominators, find the difference.

RULE 39

Apply Rule 4, Chapter 6 to Rule 31 and apply the derived rule to \( r_1 \) and \( r_2 \).

EXERCISES 39

A. \( \frac{2}{4} \) and \( \frac{1}{4} \)
B. \( \frac{3}{8} \) and \( \frac{7}{8} \)
C. \( \frac{9}{7} \) and \( \frac{3}{7} \)
D. \( \frac{1}{2} \) and \( \frac{14}{2} \)
E. \( \frac{7}{15} \) and \( \frac{12}{15} \)
\textbf{\sqrt{\text{TASK 40}}}

Given two rational numbers $r_1$ and $r_2$ represented by terminating decimals, find the difference (expressed as a decimal).

\textbf{RULE 40}

Apply Rule 4, Chapter 6 to Rule 32 and apply the derived rule to $r_1$ and $r_2$.

\textbf{EXERCISES 40}

A. 11.36 and 5.42  
B. .973 and 4.21  
C. 3.001 and .7002  
D. 16.25 and 18.73  
E. 9.3 and .7241

\textbf{\sqrt{\text{TASK 41}}}

Given two rational numbers $r_1$ and $r_2$ represented by terminating decimals, justify the algorithm for subtraction, Rule 40, by expressing the given rationals as fractions with the same denominator, subtracting the fractions, and changing the difference back to a decimal.

\textbf{RULE 41}

Apply Rule 4, Chapter 6 to Rule 33 and apply the derived rule to $r_1$ and $r_2$.

\textbf{EXERCISES 41}

A. 14.32 and 5.67  
B. 1.983 and 17.26  
C. .543 and .2176  
D. 6.03 and 2.85  
E. 1.6352 and 9.70012

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SECTION 6. Multiplication.

TASK 42

Given two rational numbers \( r_1 \) and \( r_2 \), find the product \( r_1 \times r_2 \) using the definition of multiplication of rational numbers.

RULE 42 (Definition of multiplication of rational numbers).

Choose fractional representatives \( \frac{a}{b} \) of \( r_1 \) and \( \frac{c}{d} \) of \( r_2 \). Find \( \frac{a \times c}{b \times d} \) and then find the rational number corresponding to this fraction (Rule 19).

EXAMPLES 42

A. Given: \( \left\{ \frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \ldots \right\} \) and \( \left\{ \frac{2}{5}, \frac{4}{10}, \frac{6}{15}, \frac{8}{20}, \ldots \right\} \). Answer:

Choose \( \frac{3}{9} \) and \( \frac{8}{20} \).

\[
\frac{3 \times 8}{9 \times 20} = \frac{24}{180} = \frac{3 \times 8}{9 \times 20} = \frac{4 \times 2}{3 \times 4 \times 5} = \frac{2}{15}
\]

The product is \( \left\{ \frac{2}{15}, \frac{4}{30}, \frac{6}{45}, \frac{8}{60}, \ldots \right\} \).

B. Given: \( \left\{ \frac{2}{7}, \frac{6}{14}, \frac{9}{21}, \frac{12}{28}, \ldots \right\} \) and \( \left\{ \frac{5}{3}, \frac{10}{6}, \frac{15}{9}, \frac{20}{12}, \ldots \right\} \), Answer:

Choose \( \frac{3}{7} \) and \( \frac{15}{9} \).

\[
\frac{3 \times 15}{7 \times 9} = \frac{45}{63} = \frac{5}{7}
\]

The product is \( \left\{ \frac{5}{7}, \frac{10}{14}, \frac{15}{21}, \frac{20}{28}, \ldots \right\} \).

EXERCISES 42

A. \( \left\{ \frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots \right\} \) and \( \left\{ \frac{3}{7}, \frac{6}{14}, \frac{9}{21}, \frac{12}{28}, \ldots \right\} \)

B. \( \left\{ \frac{1}{5}, \frac{2}{10}, \frac{3}{15}, \frac{4}{20}, \ldots \right\} \) and \( \left\{ \frac{5}{3}, \frac{10}{6}, \frac{15}{9}, \frac{20}{12}, \ldots \right\} \)

C. \( \left\{ \frac{6}{9}, \frac{8}{12}, \frac{10}{15}, \frac{12}{18}, \ldots \right\} \) and \( \left\{ \frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \ldots \right\} \)

D. \( \left\{ \frac{5}{7}, \frac{10}{14}, \frac{15}{21}, \frac{20}{28}, \ldots \right\} \) and \( \left\{ \frac{14}{3}, \frac{21}{6}, \frac{28}{9}, \frac{35}{12}, \ldots \right\} \)

E. \( \left\{ \frac{1}{6}, \frac{2}{12}, \frac{3}{18}, \frac{4}{24}, \ldots \right\} \) and \( \left\{ \frac{3}{5}, \frac{6}{10}, \frac{9}{15}, \frac{12}{20}, \ldots \right\} \)
Given two rational numbers \( \Sigma_1 \) and \( \Sigma_2 \), determine whether or not the closure law for multiplication of rational numbers holds for \( \Sigma_1 \) and \( \Sigma_2 \).

**RULE 43**

Apply Rule 19, Chapter 5 to Rule 11, Chapter 5 and apply the derived rule to \( \Sigma_1 \) and \( \Sigma_2 \).

**EXERCISES 43**

A. \[ \left\{ \frac{13}{5}, \frac{26}{10}, \frac{39}{15}, \frac{52}{20}, \ldots \right\} \text{ and } \left\{ \frac{5}{7}, \frac{10}{14}, \frac{15}{21}, \frac{20}{28}, \ldots \right\} \]

B. \[ \left\{ \frac{4}{5}, \frac{8}{10}, \frac{12}{15}, \frac{16}{20}, \ldots \right\} \text{ and } \left\{ \frac{1}{6}, \frac{2}{12}, \frac{3}{18}, \frac{4}{24}, \ldots \right\} \]

C. \[ \left\{ \frac{3}{8}, \frac{6}{16}, \frac{9}{24}, \frac{12}{32}, \ldots \right\} \text{ and } \left\{ \frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \frac{4}{16}, \ldots \right\} \]

D. \[ \left\{ \frac{9}{7}, \frac{18}{14}, \frac{27}{21}, \frac{36}{28}, \ldots \right\} \text{ and } \left\{ \frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots \right\} \]

E. \[ \left\{ \frac{14}{9}, \frac{28}{18}, \frac{42}{27}, \frac{56}{36}, \ldots \right\} \text{ and } \left\{ \frac{1}{5}, \frac{2}{8}, \frac{3}{12}, \frac{4}{16}, \ldots \right\} \]

Given two rational numbers \( \Sigma_1 \) and \( \Sigma_2 \), show that the operation of multiplication of rational numbers is well-defined for \( \Sigma_1 \) and \( \Sigma_2 \).

**RULE 44**

Apply Rule 23, Chapter 5 to \( \Sigma_1 \) and \( \Sigma_2 \).

**EXERCISES 44**

A. \[ \left\{ \frac{2}{4}, \frac{6}{8}, \frac{9}{12}, \frac{12}{16}, \ldots \right\} \text{ and } \left\{ \frac{4}{7}, \frac{8}{14}, \frac{12}{21}, \frac{16}{28}, \ldots \right\} \]

B. \[ \left\{ \frac{4}{3}, \frac{8}{6}, \frac{12}{9}, \frac{16}{12}, \ldots \right\} \text{ and } \left\{ \frac{15}{4}, \frac{30}{8}, \frac{45}{12}, \frac{60}{16}, \ldots \right\} \]

C. \[ \left\{ \frac{1}{6}, \frac{2}{12}, \frac{3}{18}, \frac{4}{24}, \ldots \right\} \text{ and } \left\{ \frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots \right\} \]

D. \[ \left\{ \frac{5}{8}, \frac{10}{16}, \frac{15}{24}, \frac{20}{32}, \ldots \right\} \text{ and } \left\{ \frac{1}{5}, \frac{2}{10}, \frac{3}{15}, \frac{4}{20}, \ldots \right\} \]

E. \[ \left\{ \frac{1}{6}, \frac{2}{12}, \frac{3}{18}, \frac{4}{24}, \ldots \right\} \text{ and } \left\{ \frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots \right\} \]
Given two rational numbers $r_1$ and $r_2$, construct an interpretation of the product $r_1 \times r_2$ where both numbers act as states.

**RULE 45**

Choose fractional representatives $\frac{a}{b}$ of $r_1$ and $\frac{c}{d}$ of $r_2$. Represent the state $\frac{a}{b}$ by drawing a unit rectangle, partitioning it into $b$ equal (horizontal) strips and then shading $a$ of them (add strips to be shaded if necessary). Next, represent the state $\frac{c}{d}$ by partitioning the unit rectangle into $d$ equal (vertical) strips and then shading $c$ of them (add strips to be shaded if necessary). If strips are added (either vertical or horizontal), extend the rectangles where necessary so that both are of the same size while preserving the proportion of shading. The product is represented by the intersection of the 2 shaded regions and is given by $\frac{L}{L'}$ where $L$ represents the total number of boxes in the intersection of the 2 shaded regions (doubly shaded) and $L'$ represents the total number of boxes into which the original rectangle has been divided.

**EXAMPLES 45**

A. Given: $\left\{ \frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots \right\}$ and $\left\{ \frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \frac{4}{16}, \ldots \right\}$ Answer:

Choose representatives $\frac{4}{6}$ and $\frac{1}{4}$.

The product is represented by $\frac{4}{24}$. 

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B. Given: \( \{ \frac{5}{4}, \frac{10}{8}, \frac{15}{12}, \frac{20}{16}, \ldots \} \) and \( \{ \frac{3}{2}, \frac{6}{4}, \frac{9}{6}, \frac{12}{8}, \ldots \} \). Answer:

Choose representatives \( \frac{5}{4} \) and \( \frac{3}{2} \).

The product is represented by \( \frac{15}{8} \).

**EXERCISES 45**

A. \( \{ \frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \frac{4}{16}, \ldots \} \) and \( \{ \frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots \} \)

B. \( \{ \frac{3}{2}, \frac{6}{4}, \frac{9}{6}, \frac{12}{8}, \ldots \} \) and \( \{ \frac{1}{6}, \frac{2}{12}, \frac{3}{18}, \frac{4}{24}, \ldots \} \)

C. \( \{ \frac{5}{9}, \frac{10}{6}, \frac{15}{9}, \frac{20}{12}, \ldots \} \) and \( \{ \frac{5}{4}, \frac{10}{8}, \frac{15}{12}, \frac{20}{16}, \ldots \} \)

D. \( \{ \frac{1}{5}, \frac{2}{10}, \frac{3}{15}, \frac{4}{20}, \ldots \} \) and \( \{ \frac{3}{4}, \frac{6}{8}, \frac{9}{12}, \frac{12}{16}, \ldots \} \)

E. \( \{ \frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \ldots \} \) and \( \{ \frac{5}{6}, \frac{10}{12}, \frac{15}{18}, \frac{20}{24}, \ldots \} \)

**TASK 46**

Given two rational numbers \( \Sigma_1 \) and \( \Sigma_2 \), construct an interpretation of the product \( \Sigma_1 \times \Sigma_2 \) where \( \Sigma_1 \) acts as a state and \( \Sigma_2 \) acts as an operator.
Rule 46

Choose fractional representatives $\frac{a}{b}$ of $x_1$ and $\frac{c}{d}$ of $x_2$. Represent the state $\frac{a}{b}$ as in Rule 45. Represent the operator $\frac{c}{d}$ by partitioning the state $\frac{a}{b}$ into $d$ equal strips and shading that part of the state $\frac{a}{b}$ included in $c$ of the $d$ equal strips. If strips are added (i.e., if $c > d$), extend the shaded region representing the state $\frac{a}{b}$ to the outer edges of the new rectangle formed. The product is represented by $\frac{L}{L}$ where $L$ is the number of boxes in the part just shaded and $L$ is the number of equal sized boxes into which the original rectangle has been divided.

Examples 46

A. Given $\{\frac{2}{3}, \frac{4}{6}, \frac{5}{9}, \frac{8}{12}, \ldots \}$ and $\{\frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \frac{4}{16}, \ldots \}$. Answer:

Choose representatives $\frac{4}{6}$ and $\frac{1}{4}$.

The product is represented by $\frac{4}{24}$.

B. Given $\{\frac{5}{4}, \frac{10}{8}, \frac{15}{12}, \frac{20}{16}, \ldots \}$ and $\{\frac{4}{3}, \frac{8}{6}, \frac{12}{9}, \frac{16}{12}, \ldots \}$. Answer:

Choose representatives $\frac{5}{4}$ and $\frac{4}{3}$.

The product is represented by $\frac{20}{12}$.
Chapter 7
Section 6

EXERCISES 46

A. \{\frac{5}{6}, \frac{10}{12}, \frac{15}{18}, \frac{20}{24}, \ldots \} and \{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \ldots \}

B. \{\frac{1}{6}, \frac{2}{12}, \frac{3}{18}, \frac{4}{24}, \ldots \} and \{\frac{3}{2}, \frac{6}{4}, \frac{9}{6}, \frac{12}{8}, \ldots \}

C. \{\frac{7}{6}, \frac{14}{12}, \frac{21}{18}, \frac{28}{24}, \ldots \} and \{\frac{3}{5}, \frac{6}{10}, \frac{9}{15}, \frac{12}{20}, \ldots \}

D. \{\frac{3}{2}, \frac{6}{4}, \frac{9}{6}, \frac{12}{8}, \ldots \} and \{\frac{5}{4}, \frac{10}{8}, \frac{15}{12}, \frac{20}{16}, \ldots \}

E. \{\frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \ldots \} and \{\frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \frac{4}{16}, \ldots \}

TASK 47

Given two rational numbers \(x_1\) and \(x_2\), construct an interpretation of the product \(x_1 \times x_2\) where both numbers act as operators.

RULE 47

Choose fractional representatives \(\frac{a}{b}\) of \(x_1\) and \(\frac{c}{d}\) of \(x_2\). Draw a rectangle and represent \(\frac{a}{b}\) as an operator by partitioning the rectangle into \(b\) equal sized pieces and then shading \(a\) of them. (Add strips to be shaded if necessary.) Then, let \(\frac{c}{d}\) act as an operator on the portion just chosen as in Rule 46.

EXAMPLES 47

A. Given: \{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots \} and \{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \ldots \}, Answer:

Choose representatives \(\frac{4}{6}\) and \(\frac{1}{2}\).

![Diagram of rectangle with shaded portion]
B. Given \( \{\frac{4}{3}, \frac{8}{6}, \frac{12}{9}, \frac{16}{12}, \ldots \} \) and \( \{\frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \ldots \} \), Answer:

Choose representatives \( \frac{4}{3} \) and \( \frac{2}{6} \).

The product is represented by \( \frac{8}{18} \).
EXERCISES 47

A. \(\left\{ \frac{1}{5}, \frac{2}{10}, \frac{3}{15}, \frac{4}{20}, \ldots \right\}\) and \(\left\{ \frac{3}{4}, \frac{6}{8}, \frac{9}{12}, \frac{12}{16}, \ldots \right\}\)

B. \(\left\{ \frac{7}{6}, \frac{14}{12}, \frac{21}{18}, \frac{28}{24}, \ldots \right\}\) and \(\left\{ \frac{3}{5}, \frac{6}{10}, \frac{9}{15}, \frac{12}{20}, \ldots \right\}\)

C. \(\left\{ \frac{9}{8}, \frac{18}{16}, \frac{27}{24}, \frac{36}{32}, \ldots \right\}\) and \(\left\{ \frac{4}{3}, \frac{8}{6}, \frac{12}{9}, \frac{16}{12}, \ldots \right\}\)

D. \(\left\{ \frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \frac{4}{16}, \ldots \right\}\) and \(\left\{ \frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots \right\}\)

E. \(\left\{ \frac{2}{2}, \frac{6}{4}, \frac{9}{6}, \frac{12}{8}, \ldots \right\}\) and \(\left\{ \frac{1}{6}, \frac{2}{12}, \frac{3}{18}, \frac{4}{24}, \ldots \right\}\)

TASK 48

Given two rational numbers \(x_1\) and \(x_2\) represented by terminating decimals, find the product (expressed as a decimal).

RULE 48

Multiply the numbers without regard to the decimal point. Place the decimal point in the product so that the number of digits to the right of the decimal point is the sum of the number of digits to the right of the decimal points in the two numbers being multiplied.

EXAMPLES 48

A. Given; 14.31 and 7.06. Answer:

\[
\begin{align*}
14.31 & \quad \times \quad 7.06 \\
\hline
& 8586 \\
& 1010286
\end{align*}
\]

is the product.

B. Given; 8.0432 and 15.1. Answer:

\[
\begin{align*}
8.0432 & \quad \times \quad 15.1 \\
& 402160 \\
& 80432
\end{align*}
\]

is the product.

EXERCISES 48

A. 132.3 and .56
B. 12.12 and 4.36

256
C. .985 and 2.1
D. 7.83 and 6.14
E. 21.02 and 4.001

\[ \sqrt{\text{TASK 49}} \]

Given two rational numbers \( r_1 \) and \( r_2 \) represented by terminating decimals, justify the "short-cut" algorithm for multiplication (Rule 48) by expressing each decimal as a fraction, multiplying the fractions, and changing the product back to a decimal.

\[ \text{RULE 49} \]

Apply \( \text{Rule 4, Chapter 6} \) to Rule 33 and apply the derived rule to \( r_1 \) and \( r_2 \).

\[ \text{EXERCISES 49} \]

A. 4.79 and 1.3
B. .76 and .5042
C. 18.3 and 15.17
D. 8.001 and 23.6
E. .863 and .942

\[ \sqrt{\text{TASK 50}} \]

Given two rational numbers \( r_1 \) and \( r_2 \) represented by terminating decimals, justify the algorithm for multiplication, Rule 48, by expressing the given rationals as fractions with denominators in exponential form, multiplying the fractions (keeping denominators in exponential form), and changing the product back to a decimal.

\[ \text{RULE 50} \]

(1) Multiply the decimals (Rule 48). (2) Write each decimal as a fraction (Rule 23), with denominator in the form \( 10^n \). Write the product as the product of the numerators divided by \( 10^k \) where \( k \) equals the sum of the exponents in the denominators of the two fractions. Rewrite this product by writing the numerator and placing a decimal point before the \( k \)th digit from the right. (Notice that the results of (1) and (2) are identical.)

\[ \text{EXAMPLES 50} \]

Given: 14.3 and 7.25. Answer:
14.3 can be represented as \( \frac{143}{10^1} \) and 7.25 can be represented as \( \frac{725}{10^2} \).

\[
\begin{array}{c}
143 \\
\times \frac{725}{10^2} \\
\hline
\text{1001} \\
103675
\end{array}
\]

B. Given 7.3642 and .64, Answer:

(1) 7.3642

\[
\begin{array}{c}
73642 \\
\times \frac{64}{10^2} \\
\hline
294568 \\
441852 \\
\text{4713088}
\end{array}
\]

(2) 7.3642 can be represented as \( \frac{73642}{10^4} \) and .64 can be represented as \( \frac{64}{10^2} \).

\[
\begin{array}{c}
\frac{73642}{10^4} \\
\times \frac{64}{10^2} \\
\hline
\frac{4713088}{10^6}
\end{array}
\]

EXERCISES 50

A. 7.03 and 14.2
B. 35.617 and 4.06
C. 18.92 and .6432
D. 1.5 and 2.73684
E. 12.72 and 9.13
SECTION 7. Division.

TASK 51

Given two rational numbers \( r_1 \) and \( r_2 \), find the quotient \( r_1 \div r_2 \) using the definition of division of rational numbers.

RULE 51 (Definition of division of rational numbers)

Choose fractional representatives \( \frac{a}{b} \) of \( r_1 \) and \( \frac{c}{d} \) of \( r_2 \). Find \( \frac{a \times d}{b \times c} \) and then find the rational number corresponding to this fraction (Rule 19).

EXAMPLES 51

A. Given \( \{ \frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \ldots \} \) and \( \{ \frac{5}{3}, \frac{10}{6}, \frac{15}{9}, \frac{20}{12}, \ldots \} \). Answer:

Choose \( \frac{3}{9} \) and \( \frac{20}{12} \).

\[
\frac{3 \times 12}{9 \times 20} = \frac{36}{180} = \frac{6 \times 1}{30 \times 5} = \frac{1}{5}
\]

The quotient is \( \{ \frac{1}{5}, \frac{2}{10}, \frac{3}{15}, \frac{4}{20}, \ldots \} \).

B. Given \( \{ \frac{2}{5}, \frac{4}{10}, \frac{6}{15}, \frac{8}{20}, \ldots \} \) and \( \{ \frac{3}{7}, \frac{6}{14}, \frac{9}{21}, \frac{12}{28}, \ldots \} \). Answer:

Choose \( \frac{4}{10} \) and \( \frac{3}{7} \).

\[
\frac{4 \times 7}{10 \times 3} = \frac{28}{30} = \frac{\frac{4}{5} \times 14}{\frac{3}{7} \times 15} = \frac{14}{15}
\]

The quotient is \( \{ \frac{14}{15}, \frac{28}{30}, \frac{42}{45}, \frac{56}{60}, \ldots \} \).

EXERCISES 51

A. \( \{ \frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots \} \) and \( \{ \frac{5}{3}, \frac{10}{6}, \frac{15}{9}, \frac{20}{12}, \ldots \} \)

B. \( \{ \frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \frac{4}{16}, \ldots \} \) and \( \{ \frac{3}{8}, \frac{6}{16}, \frac{9}{24}, \frac{12}{32}, \ldots \} \)

C. \( \{ \frac{7}{3}, \frac{14}{6}, \frac{21}{9}, \frac{28}{12}, \ldots \} \) and \( \{ \frac{7}{5}, \frac{14}{10}, \frac{21}{15}, \frac{28}{20}, \ldots \} \)

D. \( \{ \frac{8}{5}, \frac{16}{10}, \frac{24}{15}, \frac{32}{20}, \ldots \} \) and \( \{ \frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots \} \)

E. \( \{ \frac{3}{4}, \frac{6}{8}, \frac{9}{12}, \frac{12}{16}, \ldots \} \) and \( \{ \frac{4}{7}, \frac{8}{14}, \frac{12}{21}, \frac{16}{28}, \ldots \} \)
Chapter 7
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\textbf{Task 52}

Given a multiplication problem $\square \times x_2 = x_1$, show that division may be viewed as the inverse of multiplication by writing the corresponding division problem.

\textbf{Rule 52}

Apply Rule 35, Chapter 5 to $\square \times x_2 = x_1$.

\textbf{Exercises 52}

A. $\square \times \{ \frac{7}{8}, \frac{14}{16}, \frac{21}{24}, \frac{28}{32}, \ldots \} = \{ \frac{3}{2}, \frac{6}{4}, \frac{9}{6}, \frac{12}{8}, \ldots \}$

B. $\square \times \{ \frac{2}{3}, \frac{4}{6}, \frac{2}{8}, \frac{8}{16}, \ldots \} = \{ \frac{3}{1}, \frac{6}{2}, \frac{9}{3}, \frac{12}{4}, \ldots \}$

C. $\square \times \{ \frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \ldots \} = \{ \frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \ldots \}$

D. $\square \times \{ \frac{16}{13}, \frac{32}{26}, \frac{48}{39}, \frac{64}{52}, \ldots \} = \{ \frac{12}{5}, \frac{24}{10}, \frac{36}{15}, \frac{48}{20}, \ldots \}$

E. $\square \times \{ \frac{7}{5}, \frac{14}{10}, \frac{21}{15}, \frac{28}{20}, \ldots \} = \{ \frac{13}{7}, \frac{26}{14}, \frac{39}{21}, \frac{52}{28}, \ldots \}$

\textbf{Task 53}

Given two positive rational numbers $x_1$ and $x_2$, determine whether or not the closure law for division of positive rational numbers holds for $x_1$ and $x_2$.

\textbf{Rule 53}

Apply Rule 28, Chapter 5 to Rule 11, Chapter 5 and apply the derived rule to $x_1$ and $x_2$.

\textbf{Exercises 53}

A. $\{ \frac{4}{3}, \frac{8}{6}, \frac{12}{9}, \frac{16}{12}, \ldots \}$ and $\{ \frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \frac{4}{16}, \ldots \}$

B. $\{ \frac{3}{5}, \frac{6}{10}, \frac{9}{15}, \frac{12}{20}, \ldots \}$ and $\{ \frac{2}{7}, \frac{4}{14}, \frac{6}{21}, \frac{8}{28}, \ldots \}$

C. $\{ \frac{3}{4}, \frac{6}{8}, \frac{9}{12}, \frac{16}{16}, \ldots \}$ and $\{ \frac{1}{8}, \frac{2}{16}, \frac{3}{24}, \frac{4}{32}, \ldots \}$

D. $\{ \frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots \}$ and $\{ \frac{3}{2}, \frac{6}{4}, \frac{6}{9}, \frac{12}{8}, \ldots \}$

E. $\{ \frac{4}{5}, \frac{8}{10}, \frac{12}{15}, \frac{16}{20}, \ldots \}$ and $\{ \frac{1}{5}, \frac{2}{10}, \frac{3}{15}, \frac{4}{20}, \ldots \}$
Chapter 7  
Section 7

\( \sqrt{\text{Task 54}} \)

Given two rational numbers \( \frac{r_1}{r_2} \) and \( \frac{r_1'}{r_2'} \), show that the operation of division of rational numbers is well-defined for \( \frac{r_1}{r_2} \) and \( \frac{r_1'}{r_2'} \).

\( \text{Rule 54} \)

Apply Rule 23, Chapter 5 to \( \frac{r_1}{r_2} \) and \( \frac{r_1'}{r_2'} \).

\( \text{Exercises 54} \)

A. \( \left\{ \frac{2}{7}, \frac{4}{14}, \frac{6}{21}, \frac{8}{28} \right\} \) and \( \left\{ \frac{4}{7}, \frac{8}{14}, \frac{12}{21}, \frac{16}{28} \right\} \)

B. \( \left\{ \frac{7}{6}, \frac{14}{12}, \frac{21}{18}, \frac{28}{24} \right\} \) and \( \left\{ \frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12} \right\} \)

C. \( \left\{ \frac{5}{8}, \frac{10}{16}, \frac{15}{24}, \frac{20}{32} \right\} \) and \( \left\{ \frac{11}{4}, \frac{22}{8}, \frac{33}{12}, \frac{44}{16} \right\} \)

D. \( \left\{ \frac{10}{7}, \frac{20}{14}, \frac{30}{21}, \frac{40}{28} \right\} \) and \( \left\{ \frac{15}{4}, \frac{30}{8}, \frac{45}{12}, \frac{60}{16} \right\} \)

E. \( \left\{ \frac{1}{6}, \frac{2}{12}, \frac{3}{18}, \frac{4}{24} \right\} \) and \( \left\{ \frac{1}{7}, \frac{2}{14}, \frac{3}{21}, \frac{4}{28} \right\} \)

\( \text{Task 55} \)

Given a division problem \( \frac{a}{b} + \frac{c}{d} \), where \( \frac{a}{b} \) and \( \frac{c}{d} \) are rational numbers, show that the fraction representing the number of times the divisor \( \frac{c}{d} \) is contained in the dividend \( \frac{a}{b} \) (i.e., the quotient) is the same as the fraction obtained by applying the definition of division to the given fractions themselves.

\( \text{Rule 55} \)

(1) Find the quotient by applying the definition of division (Rule 51). (2) Construct two equal sized rectangles. Represent the first fraction, \( \frac{a}{b} \), by partitioning the first rectangle into \( b \) vertical strips and shading \( a \) of them. Represent the second fraction, \( \frac{c}{d} \), by partitioning the second rectangle into \( d \) horizontal strips and shading \( c \) of them. Next partition the first rectangle into \( d \) horizontal strips and the second rectangle into \( b \) vertical strips. (These partitions form representations of fractions equivalent to the original.) The quotient is represented by the number of shaded small rectangles \( (ad) \) in the first rectangle divided by the number of shaded (small) rectangles \( (bc) \) in the second rectangle, i.e., \( \frac{ad}{bc} \). (Notice that the results of (1) and (2) are identical.)
EXAMPLES 55

A. Given: $\frac{3}{5} + \frac{1}{2}$. Answer:

(1) $\frac{3}{5} + \frac{1}{2} = \frac{3}{5} \times \frac{2}{1} = \frac{6}{5}$.

B. Given: $\frac{1}{6} + \frac{2}{3}$. Answer:

(1) $\frac{1}{6} + \frac{2}{3} = \frac{1}{6} \times \frac{3}{2} = \frac{3}{12}$.

EXERCISES 55

A. $\frac{1}{3} + \frac{1}{4}$
B. $\frac{2}{5} + \frac{2}{3}$
C. $\frac{3}{8} + \frac{3}{4}$
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D. \( \frac{4}{5} + \frac{1}{2} \)

E. \( \frac{5}{6} + \frac{2}{3} \)

---

**TASK 56**

Given two rational numbers \( r_1 \) and \( r_2 \) represented by terminating decimals, find the quotient (expressed as a decimal).

**RULE 56**

Place a decimal point in the place reserved for the quotient \( n \) places to the right of the position of decimal point in the dividend, where \( n \) represents the number of digits appearing to the right of the decimal point in the divisor. Carry out the division as with natural numbers (Rule 30, Chapter 5).

**EXAMPLES 56**

A. Given: \( 17.3238 \div 2.6 \). Answer:

\[
\begin{array}{c|c}
2.6 & 17.3238 \\
\hline
15.6 & 15.6 \\
15.6 & \\
163 & 78 \\
156 & 78 \\
78 & \\
\end{array}
\]

B. Given: \( 49.298 \div 3.14 \). Answer:

\[
\begin{array}{c|c}
3.14 & 49.298 \\
\hline
31.4 & 17.89 \\
15.7 & 2.198 \\
15.70 & 2.198 \\
\end{array}
\]

**EXERCISES 56**

A. \( 3034 \div 22.1 \) B. \( 1190.25 \div 34.5 \) C. \( 817.8576 \div 73.023 \)

D. \( 5.1744 \div 9.24 \) E. \( 106.78 \div 1.9 \)
Given two rational numbers \( r_1 \) and \( r_2 \) represented by terminating decimals, express the quotient (of the decimals) as a quotient of whole numbers.

**RULE 57**

Write \( r_1 + r_2 = q \) and then write the corresponding multiplication problem \( q \times r_2 = r_1 \). Multiply both sides of the equation by \( 10^n \) where \( n \) denotes the larger of the number of digits appearing to the right of the decimal points in \( r_1 \) and \( r_2 \). This gives \( q \times r_2 \times 10^n = r_1 \times 10^n \). Rewrite this as \( q = \frac{r_1 \times 10^n}{r_2 \times 10^n} \).

**EXAMPLES 57**

A. Given: \( 106.78 + 1.9 \). Answer:

\[
106.78 + 1.9 = q \quad \text{so} \quad q \times 1.9 = 106.78. \quad \text{But} \quad q \times 1.9 \times 10^2 = 106.78 \times 10^2.
\]

So \( q \times 190 = 10678 \) and \( q = \frac{10678}{190} \)

B. Given: \( 5.1744 + 9.24 \). Answer:

\[
5.1744 + 9.24 = q \quad \text{so} \quad q \times 9.24 = 5.1744. \quad \text{But} \quad q \times 9.24 \times 10^4 = 5.1744 \times 10^4.
\]

So \( q = \frac{51744}{92400} \)

**EXERCISES 57**

A. \( 17.3238 + 2.6 \)
B. \( 49.298 + 3.14 \)
C. \( 566.168 + .92 \)
D. \( 654.44628 + 1.7034 \)
E. \( 102.51 + 15.3 \)
SECTION 8. Properties of the System of Positive Rationals

8.1. Closure

\textbf{Task 58}

Given two positive rationals \( r_1 \) and \( r_2 \), determine whether or not the closure property for subtraction of positive rationals holds for \( r_1 \) and \( r_2 \).

\textbf{Rule 58}

Apply Rule 19, Chapter 5 to Rule 11, Chapter 5 and apply the derived rule to \( r_1 \) and \( r_2 \).

\textbf{Exercises 58}

A. \( \left\{ \frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \ldots \right\} \) and \( \left\{ \frac{5}{4}, \frac{10}{8}, \frac{15}{12}, \frac{20}{16}, \ldots \right\} \)

B. \( \left\{ \frac{9}{8}, \frac{18}{16}, \frac{27}{24}, \frac{36}{32}, \ldots \right\} \) and \( \left\{ \frac{3}{4}, \frac{6}{8}, \frac{9}{12}, \frac{12}{16}, \ldots \right\} \)

C. \( \left\{ \frac{2}{7}, \frac{4}{14}, \frac{6}{21}, \frac{8}{28}, \ldots \right\} \) and \( \left\{ \frac{5}{4}, \frac{10}{8}, \frac{15}{12}, \frac{20}{16}, \ldots \right\} \)

D. \( \left\{ \frac{1}{7}, \frac{2}{14}, \frac{3}{21}, \frac{4}{28}, \ldots \right\} \) and \( \left\{ \frac{1}{6}, \frac{2}{12}, \frac{3}{18}, \frac{4}{24}, \ldots \right\} \)

E. \( \left\{ \frac{15}{4}, \frac{30}{8}, \frac{45}{12}, \frac{60}{16}, \ldots \right\} \) and \( \left\{ \frac{7}{3}, \frac{14}{6}, \frac{21}{9}, \frac{28}{12}, \ldots \right\} \)
Given two positive rationals $r_1$ and $r_2$, determine whether or not the commutative property for addition of positive rational numbers holds for $r_1$ and $r_2$.

**RULE 59**

Apply Rule 19, Chapter 5 to Rule 13, Chapter 5 and apply the derived rule to $r_1$ and $r_2$.

**EXERCISES 59**

A. $\left\{ \frac{1}{6}, \frac{2}{12}, \frac{3}{18}, \frac{4}{24}, \ldots \right\}$ and $\left\{ \frac{3}{5}, \frac{6}{10}, \frac{9}{15}, \frac{12}{20}, \ldots \right\}$

B. $\left\{ \frac{5}{6}, \frac{12}{18}, \frac{24}{20}, \ldots \right\}$ and $\left\{ \frac{7}{3}, \frac{14}{6}, \frac{21}{9}, \frac{28}{12}, \ldots \right\}$

C. $\left\{ \frac{1}{9}, \frac{2}{10}, \frac{3}{15}, \frac{4}{20}, \ldots \right\}$ and $\left\{ \frac{5}{3}, \frac{10}{6}, \frac{15}{9}, \frac{20}{12}, \ldots \right\}$

D. $\left\{ \frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots \right\}$ and $\left\{ \frac{3}{7}, \frac{6}{14}, \frac{9}{21}, \frac{12}{28}, \ldots \right\}$

E. $\left\{ \frac{12}{7}, \frac{24}{14}, \frac{36}{21}, \frac{48}{28}, \ldots \right\}$ and $\left\{ \frac{5}{6}, \frac{10}{12}, \frac{15}{18}, \frac{20}{24}, \ldots \right\}$

Given two positive rationals $r_1$ and $r_2$, determine whether or not the commutative property for multiplication of positive rational numbers holds for $r_1$ and $r_2$.

**RULE 60**

Apply Rule 19, Chapter 5 to Rule 13, Chapter 5 and apply the derived rule to $r_1$ and $r_2$.

**EXERCISES 60**

A. $\left\{ \frac{1}{6}, \frac{2}{12}, \frac{3}{18}, \frac{4}{24}, \ldots \right\}$ and $\left\{ \frac{3}{5}, \frac{6}{10}, \frac{9}{15}, \frac{12}{20}, \ldots \right\}$

B. $\left\{ \frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots \right\}$ and $\left\{ \frac{5}{6}, \frac{10}{12}, \frac{15}{18}, \frac{20}{24}, \ldots \right\}$

C. $\left\{ \frac{10}{7}, \frac{20}{14}, \frac{30}{21}, \frac{40}{28}, \ldots \right\}$ and $\left\{ \frac{2}{5}, \frac{4}{10}, \frac{6}{15}, \frac{8}{20}, \ldots \right\}$

D. $\left\{ \frac{4}{3}, \frac{8}{6}, \frac{12}{9}, \frac{16}{12}, \ldots \right\}$ and $\left\{ \frac{3}{2}, \frac{6}{4}, \frac{9}{6}, \frac{12}{8}, \ldots \right\}$

E. $\left\{ \frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \ldots \right\}$ and $\left\{ \frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \ldots \right\}$
Given three positive rationals \( \mathbb{R}_1, \mathbb{R}_2 \) and \( \mathbb{R}_3 \), determine whether or not the associative property for addition of positive rational numbers holds for \( \mathbb{R}_1, \mathbb{R}_2 \) and \( \mathbb{R}_3 \).

**RULE 61**

Apply Rule 19, Chapter 5 to Rule 12, Chapter 5 and apply the derived rule to \( \mathbb{R}_1, \mathbb{R}_2 \) and \( \mathbb{R}_3 \).

**EXERCISES 61**

A. \( \left\{ \frac{1}{8}, \frac{2}{16}, \frac{3}{24}, \frac{4}{32}, \ldots \right\} \), \( \left\{ \frac{5}{10}, \frac{10}{15}, \frac{15}{20}, \frac{20}{24}, \ldots \right\} \), \( \left\{ \frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \ldots \right\} \)

B. \( \left\{ \frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots \right\} \), \( \left\{ \frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \frac{4}{16}, \ldots \right\} \), \( \left\{ \frac{5}{10}, \frac{10}{15}, \frac{15}{20}, \frac{20}{24}, \ldots \right\} \)

C. \( \left\{ \frac{7}{5}, \frac{14}{10}, \frac{21}{15}, \frac{28}{20}, \ldots \right\} \), \( \left\{ \frac{3}{2}, \frac{6}{4}, \frac{9}{6}, \frac{12}{8}, \ldots \right\} \), \( \left\{ \frac{4}{5}, \frac{8}{10}, \frac{12}{15}, \frac{16}{20}, \ldots \right\} \)

D. \( \left\{ \frac{4}{7}, \frac{8}{14}, \frac{12}{21}, \frac{16}{28}, \ldots \right\} \), \( \left\{ \frac{3}{7}, \frac{6}{14}, \frac{9}{21}, \frac{12}{28}, \ldots \right\} \), \( \left\{ \frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \ldots \right\} \)

E. \( \left\{ \frac{8}{3}, \frac{16}{6}, \frac{24}{9}, \frac{32}{12}, \ldots \right\} \), \( \left\{ \frac{3}{4}, \frac{6}{8}, \frac{9}{12}, \frac{12}{16}, \ldots \right\} \), \( \left\{ \frac{7}{6}, \frac{14}{12}, \frac{21}{18}, \frac{28}{24}, \ldots \right\} \)

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**TASK 62**

Given three positive rationals \( \mathbb{R}_1, \mathbb{R}_2 \) and \( \mathbb{R}_3 \), determine whether or not the associative property for multiplication of positive rational numbers holds for \( \mathbb{R}_1, \mathbb{R}_2 \) and \( \mathbb{R}_3 \).

**RULE 62**

Apply Rule 19, Chapter 5 to Rule 12, Chapter 5 and apply the derived rule to \( \mathbb{R}_1, \mathbb{R}_2 \) and \( \mathbb{R}_3 \).

**EXERCISES 62**

A. \( \left\{ \frac{2}{2}, \frac{6}{4}, \frac{9}{6}, \frac{12}{8}, \ldots \right\} \), \( \left\{ \frac{5}{3}, \frac{8}{10}, \frac{12}{15}, \frac{16}{20}, \ldots \right\} \), \( \left\{ \frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \ldots \right\} \)

B. \( \left\{ \frac{7}{4}, \frac{14}{8}, \frac{21}{12}, \frac{28}{16}, \ldots \right\} \), \( \left\{ \frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \ldots \right\} \), \( \left\{ \frac{8}{9}, \frac{16}{18}, \frac{24}{27}, \frac{32}{36}, \ldots \right\} \)

C. \( \left\{ \frac{5}{6}, \frac{10}{12}, \frac{15}{18}, \frac{20}{24}, \ldots \right\} \), \( \left\{ \frac{3}{8}, \frac{6}{16}, \frac{9}{24}, \frac{12}{32}, \ldots \right\} \), \( \left\{ \frac{8}{7}, \frac{16}{14}, \frac{24}{21}, \frac{32}{28}, \ldots \right\} \)

D. \( \left\{ \frac{1}{6}, \frac{2}{12}, \frac{3}{18}, \frac{4}{24}, \ldots \right\} \), \( \left\{ \frac{3}{2}, \frac{6}{4}, \frac{9}{6}, \frac{12}{8}, \ldots \right\} \), \( \left\{ \frac{4}{11}, \frac{8}{22}, \frac{12}{33}, \frac{16}{44}, \ldots \right\} \)

E. \( \left\{ \frac{9}{5}, \frac{18}{10}, \frac{27}{15}, \frac{36}{20}, \ldots \right\} \), \( \left\{ \frac{10}{3}, \frac{20}{6}, \frac{30}{9}, \frac{40}{12}, \ldots \right\} \), \( \left\{ \frac{2}{8}, \frac{6}{16}, \frac{9}{24}, \frac{12}{32}, \ldots \right\} \)


\textbf{TASK 63}

Given two positive rationals \( r_1 \) and \( r_2 \), determine whether or not the commutative property for subtraction of positive rational numbers holds for \( r_1 \) and \( r_2 \).

\textbf{RULE 63}

Apply \(^*\) Rule 19, Chapter 5 to Rule 13, Chapter 5 and then apply the derived rule to \( r_1 \) and \( r_2 \).

\textbf{EXERCISES 63}

A. \( \left\{ \frac{7}{3}, \frac{14}{6}, \frac{21}{9}, \frac{28}{12}, \ldots \right\} \) and \( \left\{ \frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \ldots \right\} \)

B. \( \left\{ \frac{3}{4}, \frac{6}{8}, \frac{9}{12}, \frac{12}{16}, \ldots \right\} \) and \( \left\{ \frac{7}{8}, \frac{14}{16}, \frac{21}{24}, \frac{28}{32}, \ldots \right\} \)

C. \( \left\{ \frac{6}{5}, \frac{12}{10}, \frac{18}{15}, \frac{24}{20}, \ldots \right\} \) and \( \left\{ \frac{7}{3}, \frac{14}{6}, \frac{21}{9}, \frac{28}{12}, \ldots \right\} \)

D. \( \left\{ \frac{3}{8}, \frac{6}{16}, \frac{9}{24}, \frac{12}{32}, \ldots \right\} \) and \( \left\{ \frac{1}{6}, \frac{2}{12}, \frac{3}{18}, \frac{4}{24}, \ldots \right\} \)

E. \( \left\{ \frac{5}{9}, \frac{10}{15}, \frac{15}{20}, \ldots \right\} \) and \( \left\{ \frac{2}{7}, \frac{4}{14}, \frac{6}{21}, \frac{8}{28}, \ldots \right\} \)

\textbf{TASK 64}

Given three positive rationals \( r_1 \), \( r_2 \) and \( r_3 \), determine whether or not the associative property for subtraction of the positive rational numbers holds for \( r_1 \), \( r_2 \) and \( r_3 \).

\textbf{RULE 64}

Apply \(^*\) Rule 19, Chapter 5 to Rule 12, Chapter 5 and apply the derived rule to \( r_1 \), \( r_2 \) and \( r_3 \).

\textbf{EXERCISES 64}

A. \( \left\{ \frac{3}{2}, \frac{6}{4}, \frac{9}{6}, \frac{12}{8}, \ldots \right\} \), \( \left\{ \frac{4}{5}, \frac{8}{10}, \frac{12}{15}, \frac{16}{20}, \ldots \right\} \), \( \left\{ \frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \ldots \right\} \)

B. \( \left\{ \frac{7}{4}, \frac{14}{8}, \frac{21}{12}, \frac{28}{16}, \ldots \right\} \), \( \left\{ \frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \ldots \right\} \), \( \left\{ \frac{8}{9}, \frac{16}{18}, \frac{24}{27}, \frac{32}{36}, \ldots \right\} \)

C. \( \left\{ \frac{6}{7}, \frac{16}{14}, \frac{24}{21}, \frac{32}{28}, \ldots \right\} \), \( \left\{ \frac{5}{6}, \frac{10}{12}, \frac{15}{18}, \frac{20}{24}, \ldots \right\} \), \( \left\{ \frac{1}{8}, \frac{2}{16}, \frac{3}{24}, \frac{4}{32}, \ldots \right\} \)

D. \( \left\{ \frac{1}{6}, \frac{2}{12}, \frac{3}{18}, \frac{4}{24}, \ldots \right\} \), \( \left\{ \frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \ldots \right\} \), \( \left\{ \frac{8}{9}, \frac{16}{18}, \frac{24}{27}, \frac{32}{36}, \ldots \right\} \)

E. \( \left\{ \frac{10}{3}, \frac{20}{6}, \frac{30}{9}, \frac{40}{12}, \ldots \right\} \), \( \left\{ \frac{9}{5}, \frac{18}{10}, \frac{27}{15}, \frac{36}{20}, \ldots \right\} \), \( \left\{ \frac{3}{8}, \frac{6}{16}, \frac{9}{24}, \frac{12}{32}, \ldots \right\} \)
Given two positive rationals \( r_1 \) and \( r_2 \), determine whether or not the commutative property for division of positive rational numbers holds for \( r_1 \) and \( r_2 \).

**RULE 65**

Apply Rule 19, Chapter 5 to Rule 13, Chapter 5 and apply the derived rule to \( r_1 \) and \( r_2 \).

**EXERCISES 65**

A. \[ \left\{ \frac{15}{7}, \frac{30}{14}, \frac{45}{21}, \frac{60}{28}, \ldots \right\} \text{ and } \left\{ \frac{3}{8}, \frac{6}{16}, \frac{9}{24}, \frac{12}{32}, \ldots \right\} \]

B. \[ \left\{ \frac{2}{4}, \frac{5}{6}, \frac{8}{12}, \ldots \right\} \text{ and } \left\{ \frac{11}{22}, \frac{22}{44}, \frac{33}{66}, \frac{44}{88}, \ldots \right\} \]

C. \[ \left\{ \frac{5}{9}, \frac{10}{18}, \frac{15}{27}, \frac{20}{36}, \ldots \right\} \text{ and } \left\{ \frac{11}{22}, \frac{22}{44}, \frac{33}{66}, \frac{44}{88}, \ldots \right\} \]

D. \[ \left\{ \frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \frac{4}{16}, \ldots \right\} \text{ and } \left\{ \frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \ldots \right\} \]

E. \[ \left\{ \frac{11}{3}, \frac{22}{6}, \frac{33}{9}, \frac{44}{12}, \ldots \right\} \text{ and } \left\{ \frac{4}{7}, \frac{8}{14}, \frac{12}{21}, \frac{16}{28}, \ldots \right\} \]

Given three positive rationals \( r_1, r_2 \), and \( r_3 \), determine whether or not the associative property for division of positive rational numbers holds for \( r_1, r_2 \), and \( r_3 \).

**RULE 66**

Apply Rule 19, Chapter 5 to Rule 12, Chapter 5 and apply the derived rule to \( r_1, r_2 \), and \( r_3 \).

**EXERCISES 66**

A. \[ \left\{ \frac{1}{9}, \frac{2}{18}, \frac{3}{27}, \frac{4}{36}, \ldots \right\}, \left\{ \frac{2}{4}, \frac{6}{8}, \frac{9}{12}, \frac{12}{16}, \ldots \right\}, \left\{ \frac{5}{6}, \frac{10}{12}, \frac{15}{18}, \frac{20}{24}, \ldots \right\} \]

B. \[ \left\{ \frac{4}{7}, \frac{8}{14}, \frac{12}{21}, \frac{16}{28}, \ldots \right\}, \left\{ \frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \ldots \right\}, \left\{ \frac{14}{5}, \frac{28}{10}, \frac{42}{15}, \frac{56}{20}, \ldots \right\} \]

C. \[ \left\{ \frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \frac{4}{16}, \ldots \right\}, \left\{ \frac{14}{9}, \frac{28}{18}, \frac{42}{27}, \frac{56}{36}, \ldots \right\}, \left\{ \frac{6}{7}, \frac{12}{14}, \frac{18}{21}, \frac{24}{28}, \ldots \right\} \]

D. \[ \left\{ \frac{3}{5}, \frac{6}{10}, \frac{9}{15}, \frac{12}{20}, \ldots \right\}, \left\{ \frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots \right\}, \left\{ \frac{5}{7}, \frac{10}{14}, \frac{15}{21}, \frac{20}{28}, \ldots \right\} \]

E. \[ \left\{ \frac{12}{7}, \frac{24}{14}, \frac{36}{21}, \frac{48}{28}, \ldots \right\}, \left\{ \frac{14}{3}, \frac{28}{6}, \frac{42}{9}, \frac{56}{12}, \ldots \right\}, \left\{ \frac{6}{1}, \frac{12}{2}, \frac{18}{3}, \frac{24}{4}, \ldots \right\} \]
8.3. Identity.

\[ \text{\textsc{task 67}} \]

Given a non-negative rational number \( r \), determine whether or not \( \{0, 0, 0, 0, \ldots\} \) is an additive identity for \( r \).

\[ \text{\textsc{rule 67}} \]

Apply Rule 19, Chapter 5 to Rule 14, Chapter 5 and apply the derived rule to \( r \).

\[ \text{\textsc{exercises 67}} \]

A. \( \{\frac{1}{6}, \frac{2}{12}, \frac{3}{18}, \frac{4}{24}, \ldots\} \)
B. \( \{\frac{5}{6}, \frac{12}{10}, \frac{18}{15}, \frac{24}{20}, \ldots\} \)
C. \( \{\frac{1}{5}, \frac{2}{10}, \frac{3}{15}, \frac{4}{20}, \ldots\} \)
D. \( \{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots\} \)
E. \( \{\frac{12}{7}, \frac{24}{14}, \frac{36}{21}, \frac{48}{28}, \ldots\} \)

(Note: \( \{0, 0, 0, 0, \ldots\} \) is an additive identity for all non-negative rationals.)

\[ \text{\textsc{task 68}} \]

Given a positive rational number \( r \), determine whether or not \( \{\frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \frac{4}{4}, \ldots\} \) is a multiplicative identity for \( r \).

\[ \text{\textsc{rule 68}} \]

Apply Rule 19, Chapter 5 to Rule 14, Chapter 5 and apply the derived rule to \( r \).

\[ \text{\textsc{exercises 68}} \]

A. \( \{\frac{5}{6}, \frac{10}{12}, \frac{15}{18}, \frac{20}{24}, \ldots\} \)
B. \( \{\frac{1}{4}, \frac{2}{2}, \frac{3}{12}, \frac{4}{16}, \ldots\} \)
C. \( \{\frac{2}{2}, \frac{6}{4}, \frac{2}{12}, \frac{12}{8}, \ldots\} \)
D. \( \{\frac{5}{7}, \frac{10}{14}, \frac{15}{21}, \frac{20}{28}, \ldots\} \)
E. \( \{\frac{2}{4}, \frac{6}{8}, \frac{9}{12}, \frac{12}{16}, \ldots\} \)

(Note: \( \{\frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \frac{4}{4}, \ldots\} \) is a multiplicative identity for all positive rationals.)
8.4. **Inverse.**

**TASK 69**

Given two positive rational numbers \( r_2 \) and \( r_3 \), and the identity \( \Xi_1 = \left\{ \frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \ldots \right\} \) for multiplication of positive rational numbers, determine whether or not \( r_2 \) and \( r_3 \) are inverses under multiplication.

**RULE 69**

Apply Rule 19, Chapter 5 to Rule 15, Chapter 5 and apply the derived rule to \( \Xi_1, \Xi_2 \) and \( \Xi_3 \).

**EXERCISES 69**

A. \( \left\{ \frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots \right\}, \left\{ \frac{3}{2}, \frac{6}{4}, \frac{9}{6}, \frac{12}{8}, \ldots \right\} \)  
B. \( \left\{ \frac{5}{12}, \frac{10}{24}, \frac{15}{36}, \frac{20}{48}, \ldots \right\}, \left\{ \frac{12}{5}, \frac{24}{10}, \frac{36}{15}, \frac{48}{20}, \ldots \right\} \)  
C. \( \left\{ \frac{4}{5}, \frac{8}{10}, \frac{12}{15}, \frac{16}{20}, \ldots \right\}, \left\{ \frac{5}{4}, \frac{10}{8}, \frac{15}{12}, \frac{20}{16}, \ldots \right\} \)  
D. \( \left\{ \frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \ldots \right\}, \left\{ \frac{2}{1}, \frac{4}{2}, \frac{6}{3}, \frac{8}{4}, \ldots \right\} \)  
E. \( \left\{ \frac{7}{6}, \frac{14}{12}, \frac{21}{18}, \frac{28}{24}, \ldots \right\}, \left\{ \frac{6}{7}, \frac{12}{14}, \frac{18}{21}, \frac{24}{28}, \ldots \right\} \)
8.5. Distributivity.

\[ \text{TASK 70} \]

Given three positive rational numbers \( \xi_1, \xi_2, \) and \( \xi_3, \) determine whether or not the distributive property of multiplication over addition holds for \( \xi_1, \xi_2, \) and \( \xi_3. \)

\[ \text{RULE 70} \]

Apply \( \text{Rule 19, Chapter 5 to Rule 36, Chapter 5} \) and apply the derived rule to \( \xi_1, \xi_2, \) and \( \xi_3. \)

\[ \text{EXERCISES 70} \]

A. \( \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots \}, \{\frac{3}{4}, \frac{5}{6}, \frac{9}{12}, \frac{12}{16}, \ldots \}, \{\frac{5}{6}, \frac{10}{12}, \frac{15}{18}, \frac{20}{24}, \ldots \} \)

B. \( \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots \}, \{\frac{3}{4}, \frac{5}{6}, \frac{9}{12}, \frac{12}{16}, \ldots \}, \{\frac{5}{6}, \frac{10}{12}, \frac{15}{18}, \frac{20}{24}, \ldots \} \)

C. \( \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots \}, \{\frac{3}{4}, \frac{5}{6}, \frac{9}{12}, \frac{12}{16}, \ldots \}, \{\frac{5}{6}, \frac{10}{12}, \frac{15}{18}, \frac{20}{24}, \ldots \} \)

D. \( \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots \}, \{\frac{3}{4}, \frac{5}{6}, \frac{9}{12}, \frac{12}{16}, \ldots \}, \{\frac{5}{6}, \frac{10}{12}, \frac{15}{18}, \frac{20}{24}, \ldots \} \)

E. \( \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots \}, \{\frac{3}{4}, \frac{5}{6}, \frac{9}{12}, \frac{12}{16}, \ldots \}, \{\frac{5}{6}, \frac{10}{12}, \frac{15}{18}, \frac{20}{24}, \ldots \} \)

\[ \text{TASK 71} \]

Given three positive rational numbers \( \xi_1, \xi_2, \) and \( \xi_3, \) determine whether or not the distributive property of multiplication over subtraction (when the subtraction can be performed) holds for \( \xi_1, \xi_2, \) and \( \xi_3. \)

\[ \text{RULE 71} \]

Apply \( \text{Rule 19, Chapter 5 to Rule 36, Chapter 5} \) and apply the derived rule to \( \xi_1, \xi_2, \) and \( \xi_3. \)

\[ \text{EXERCISES 71} \]

A. \( \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots \}, \{\frac{4}{5}, \frac{8}{10}, \frac{12}{15}, \frac{16}{20}, \ldots \}, \{\frac{5}{6}, \frac{10}{12}, \frac{15}{18}, \frac{20}{24}, \ldots \} \)

B. \( \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots \}, \{\frac{4}{5}, \frac{8}{10}, \frac{12}{15}, \frac{16}{20}, \ldots \}, \{\frac{5}{6}, \frac{10}{12}, \frac{15}{18}, \frac{20}{24}, \ldots \} \)

C. \( \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots \}, \{\frac{4}{5}, \frac{8}{10}, \frac{12}{15}, \frac{16}{20}, \ldots \}, \{\frac{5}{6}, \frac{10}{12}, \frac{15}{18}, \frac{20}{24}, \ldots \} \)

D. \( \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots \}, \{\frac{4}{5}, \frac{8}{10}, \frac{12}{15}, \frac{16}{20}, \ldots \}, \{\frac{5}{6}, \frac{10}{12}, \frac{15}{18}, \frac{20}{24}, \ldots \} \)

E. \( \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots \}, \{\frac{4}{5}, \frac{8}{10}, \frac{12}{15}, \frac{16}{20}, \ldots \}, \{\frac{5}{6}, \frac{10}{12}, \frac{15}{18}, \frac{20}{24}, \ldots \} \)
8.6. Other Properties: Relationships Between the Natural Numbers and the Positive Rationals

TASK 72

Given two natural numbers $k$ and $m$, an operation $\ast$, and the correspondence $n \leftrightarrow \{\frac{n}{1}, \frac{2n}{2}, \frac{3n}{3}, \frac{4n}{4}, \ldots \}$ between natural numbers and positive rational numbers, show that $k \ast m \leftrightarrow \{\frac{k}{1}, \frac{2k}{2}, \frac{3k}{3}, \frac{4k}{4}, \ldots \} \ast \{\frac{m}{1}, \frac{2m}{2}, \frac{3m}{3}, \frac{4m}{4}, \ldots \}$.

RULE 72

Find $k \ast m$ using the appropriate rule for natural numbers. Next find $\{\frac{k}{1}, \frac{2k}{2}, \frac{3k}{3}, \frac{4k}{4}, \ldots \} \ast \{\frac{m}{1}, \frac{2m}{2}, \frac{3m}{3}, \frac{4m}{4}, \ldots \}$ using the appropriate rule for positive rationals. Observe that $k \ast m \leftrightarrow \{\frac{k}{1}, \frac{2k}{2}, \frac{3k}{3}, \frac{4k}{4}, \ldots \} \ast \{\frac{m}{1}, \frac{2m}{2}, \frac{3m}{3}, \frac{4m}{4}, \ldots \} = \{\frac{k \ast m}{1}, \frac{2(k \ast m)}{2}, \frac{3(k \ast m)}{3}, \frac{4(k \ast m)}{4}, \ldots \}$.

EXAMPLES 72

A. Given: 4, 6 and +. Answer:

\[ 4 + 6 = 10. \]

\[ \{\frac{4}{1}, \frac{8}{2}, \frac{12}{3}, \frac{16}{4}, \ldots \} + \{\frac{6}{1}, \frac{12}{2}, \frac{18}{3}, \frac{24}{4}, \ldots \} = \{\frac{10}{1}, \frac{20}{2}, \frac{30}{3}, \frac{40}{4}, \ldots \}. \]

Observe that $4 + 6 = 10 \leftrightarrow \{\frac{4}{1}, \frac{8}{2}, \frac{12}{3}, \frac{16}{4}, \ldots \} + \{\frac{6}{1}, \frac{12}{2}, \frac{18}{3}, \frac{24}{4}, \ldots \} = \{\frac{10}{1}, \frac{20}{2}, \frac{30}{3}, \frac{40}{4}, \ldots \}$.

B. Given: 3, 5 and x. Answer:

\[ 3 \times 5 = 15. \]

\[ \{\frac{3}{1}, \frac{6}{2}, \frac{9}{3}, \frac{12}{4}, \ldots \} \times \{\frac{5}{1}, \frac{10}{2}, \frac{15}{3}, \frac{20}{4}, \ldots \} = \{\frac{15}{1}, \frac{30}{2}, \frac{45}{3}, \frac{60}{4}, \ldots \}. \]

Observe that $3 \times 5 = 15 \leftrightarrow \{\frac{3}{1}, \frac{6}{2}, \frac{9}{3}, \frac{12}{4}, \ldots \} \times \{\frac{5}{1}, \frac{10}{2}, \frac{15}{3}, \frac{20}{4}, \ldots \} = \{\frac{15}{1}, \frac{30}{2}, \frac{45}{3}, \frac{60}{4}, \ldots \}$.

EXERCISES 72

A. 3, 7 and +
B. 6, 2 and +
C. 5, 9 and +
D. 8, 2 and -
E. 9, 3 and x

TASK 73

Construct a 1-1 correspondence between the natural numbers and positive rational numbers.

RULE 73

Represent each rational by its canonical form (Rule 20). Construct a rectangular array of fractions so that each fraction in the first row has numerator 1, in the second row numerator 2, in the third row numerator 3, etc. Never repeat a fraction (canonical form) which has been previously listed. Pair the element in the upper left hand corner of the array with the element to the right with 2 and then proceed along the diagonal to the left and downward, pairing each element in turn. When the first column is reached, go one row down and proceed along the diagonal to the right and upward until you reach the first row. Pair the next element to the right and continue the process as before.

EXAMPLE 73

\[
\begin{array}{cccccccc}
\frac{1}{1} & \rightarrow & \frac{1}{2} & \rightarrow & \frac{1}{3} & \rightarrow & \frac{1}{4} & \rightarrow & \cdots & \frac{1}{n} & \cdots \\
1 & & 2 & & 3 & & 4 & & \cdots & n & \cdots \\
\frac{2}{1} & \rightarrow & \frac{2}{3} & \rightarrow & \frac{2}{5} & \rightarrow & \frac{2}{7} & \rightarrow & \cdots & \frac{2}{n} & \cdots \\
1 & & 3 & & 5 & & 7 & & \cdots & n & \cdots \\
\frac{3}{1} & \rightarrow & \frac{3}{2} & \rightarrow & \frac{3}{4} & \rightarrow & \frac{3}{5} & \rightarrow & \cdots & \frac{3}{n} & \cdots \\
1 & & 2 & & 4 & & 5 & & \cdots & n & \cdots \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\frac{m}{1} & \rightarrow & \frac{m}{2} & \rightarrow & \frac{m}{3} & \rightarrow & \frac{m}{4} & \rightarrow & \cdots & \frac{m}{n} & \cdots \\
1 & \leftrightarrow & \frac{1}{1} & \leftrightarrow & \frac{2}{3} & \leftrightarrow & \frac{2}{5} & \leftrightarrow & \cdots & \frac{9}{2} & \leftrightarrow & \frac{3}{2} \\
2 & \leftrightarrow & \frac{1}{2} & \leftrightarrow & \frac{1}{3} & \leftrightarrow & \frac{1}{4} & \leftrightarrow & \cdots & \frac{6}{2} & \leftrightarrow & \frac{1}{2} \\
3 & \leftrightarrow & \frac{2}{1} & \leftrightarrow & \frac{1}{4} & \leftrightarrow & \frac{1}{5} & \leftrightarrow & \cdots & \frac{7}{2} & \leftrightarrow & \frac{1}{4} \\
4 & \leftrightarrow & \frac{3}{1} & \leftrightarrow & \frac{2}{5} & \leftrightarrow & \frac{2}{6} & \leftrightarrow & \cdots & \frac{8}{2} & \leftrightarrow & \frac{2}{5} \\
\end{array}
\]
EXERCISES 73

Construct a different 1-1 correspondence between the natural numbers and the positive rational numbers.

TASK 74

Given two positive rationals \( \mathbf{x}_1 \) and \( \mathbf{x}_2 \), show that there is a positive rational between them.

RULE 74

Find the positive rational \((\mathbf{x}_1 + \mathbf{x}_2) + 2\) by using Rules 27 and 51. Order \( \mathbf{x}_1 \), \( \mathbf{x}_2 \) and \((\mathbf{x}_1 + \mathbf{x}_2) + 2\) by using Rule 22.

EXAMPLES 74

A. Given: \( \mathbf{x}_1 = \{7, 14, 21, 28, 35, \ldots \} \) and \( \mathbf{x}_2 = \{3, 6, 9, 12, \ldots \} \).

Answer:

\[
\mathbf{x}_1 + \mathbf{x}_2 = \left\{ \frac{71}{10}, \frac{142}{20}, \frac{213}{30}, \frac{284}{40}, \ldots \right\} \quad \text{(Rule 27),}
\]

\[
(\mathbf{x}_1 + \mathbf{x}_2) + 2 = \left\{ \frac{71}{80}, \frac{162}{160}, \frac{213}{240}, \frac{284}{320}, \ldots \right\} \quad \text{(Rules 27 and 51),}
\]

\[
\mathbf{x}_1 > \left\{ \frac{71}{80}, \frac{162}{160}, \frac{213}{240}, \frac{284}{320}, \ldots \right\} > \mathbf{x}_2, \quad \text{because} \quad \frac{112}{80} > \frac{71}{80} > \frac{30}{80} \quad \text{by Rule 22.}
\]

B. Given: \( \mathbf{x}_1 = \{15, 30, 45, 60, 75, \ldots \} \) and \( \mathbf{x}_2 = \{7, 14, 21, 28, 35, \ldots \} \).

Answer:

\[
\mathbf{x}_1 + \mathbf{x}_2 = \left\{ \frac{29}{8}, \frac{58}{16}, \frac{87}{24}, \frac{116}{32}, \ldots \right\} \quad \text{(Rule 27),}
\]

\[
(\mathbf{x}_1 + \mathbf{x}_2) + 2 = \left\{ \frac{29}{16}, \frac{58}{32}, \frac{87}{48}, \frac{116}{64}, \ldots \right\} \quad \text{(Rules 27 and 51),}
\]

\[
\mathbf{x}_1 > \left\{ \frac{29}{16}, \frac{58}{32}, \frac{87}{48}, \frac{116}{64}, \ldots \right\} > \mathbf{x}_2, \quad \text{(Rule 22) because} \quad \frac{30}{16} > \frac{29}{16} > \frac{28}{16}.
\]

EXERCISES 74

A. \( \{\frac{1}{3}, 2, 3, 4, \ldots \} \) and \( \{9, 18, 27, 36, \ldots \} \)
TASK 75

Given any positive rational $r$, show that there is a positive rational smaller than $r$.

RULE 75

Find $r + 2$ (Rule 51). Apply Rule 22 to show this quotient is smaller than $r$.

EXAMPLES 75

A. Given: $r = \{\frac{1}{15}, \frac{2}{30}, \frac{3}{45}, \frac{4}{60}, \ldots \}$.

Answer:

$r + 2 = \{\frac{1}{30}, \frac{2}{60}, \frac{3}{90}, \frac{4}{120}, \ldots \}$ (Rule 51).

$r > \{\frac{1}{30}, \frac{2}{60}, \frac{3}{90}, \frac{4}{120}, \ldots \}$ (Rule 22).

B. Given: $r = \{\frac{1}{1000}, \frac{2}{2000}, \frac{3}{3000}, \frac{4}{4000}, \ldots \}$

Answer:

$r + 2 = \{\frac{1}{2000}, \frac{2}{4000}, \frac{3}{6000}, \frac{4}{8000}, \ldots \}$ (Rule 51).

$r > \{\frac{1}{2000}, \frac{2}{4000}, \frac{3}{6000}, \frac{4}{8000}, \ldots \}$ (Rule 22).

EXERCISES 75

A. $\{\frac{2}{35}, \frac{4}{70}, \frac{6}{105}, \frac{8}{140}, \ldots \}$

B. $\{\frac{3}{700}, \frac{6}{1400}, \frac{9}{2100}, \frac{12}{2800}, \ldots \}$
C. \( \left\{ \frac{1}{20}, \frac{2}{40}, \frac{3}{60}, \frac{4}{80}, \ldots \right\} \)
D. \( \left\{ \frac{1}{10000}, \frac{2}{20000}, \frac{3}{30000}, \frac{4}{40000}, \ldots \right\} \)
E. \( \left\{ \frac{2}{15}, \frac{4}{30}, \frac{6}{45}, \frac{8}{60}, \ldots \right\} \)
SECTION 1. **More-Less Property**

**Task 1**

Given an ordered pair of sets, state its "more-less" property.

**Rule 1**

Count the number of elements in the first set (say \(a\)) and the number of elements in the second set (say \(b\)). Find the difference \(d\) of \(a\) and \(b\) (larger-smaller). If \(a\) is greater than \(b\), then \(a\) is \(d\) more than \(b\). If \(a\) is less than \(b\), then \(a\) is \(d\) less than \(b\).

**Examples 1**

A. Given: \([x, y, z], [1, 2]\).
   Answer:
   The first set \([x, y, z]\) has 3 elements and the second set \([1, 2]\) has 2 elements. \(3 - 2 = 1\). \(3\) is 1 more than 2.

B. Given: \([1], [5, 7, b, g]\).
   Answer:
   The first set \([1]\) has 1 element and the second set \([5, 7, b, g]\) has 4 elements. \(4 - 1 = 3\). 1 is 3 less than 4.

**Exercises 1**

A. \([5], [4, 7, a]\)
B. \([3, 4, 7], [x]\)
C. \([h, 1, p, z], [1, 5, 9]\)
D. \([2, 5, 6, 7], [1, 4, 8, 3, 6]\)
E. \([a, b], [a, b, c]\)
TASK 2

Given an ordered pair \((a, b)\) of whole numbers, state the "more-less" property it represents.

RULE 2

Apply the latter part of Rule 1.

EXAMPLES 2

A. Given: \((4, 2)\).
   Answer: 
   \[4 - 2 = 2. \text{ 4 is 2 more than 2.}\]

B. Given: \((3, 7)\).
   Answer: 
   \[7 - 3 = 4. \text{ 3 is 4 less than 7.}\]

EXERCISES 2

A. \((16, 23)\)
B. \((14, 11)\)
C. \((5, 24)\)
D. \((9, 8)\)
E. \((12, 18)\)

TASK 3

Given the "more-less" property, "\(a\) is \(n\) less (\(n\) more) than \(b\)," construct the equivalence class it defines.

RULE 3

Construct a set whose elements are ordered pairs of sets in which the number of elements in the first (second) set of the first pair is zero, in the first (second) set of the second pair is 1, in the first (second) set of the third pair is 2, etc. The number of elements in the second (first) set of each pair is \(n\) more than the number of elements in the first (second) set (of the same pair).

EXAMPLES 3

A. Given: 2 is 3 less than 5.
   Construct:
   \[
   \{([\   ], [1, 2, 3]), ([a], [a, b, c, d]), ([1, 2], [1, 2, 3, 4, 5]),
   ([a, b, c], [1, 2, 3, 4, 5, 6]), \ldots \}
   \]
B. Given: 5 is 1 more than 4.
Construct:
\[ \{\{a\}, \{\}\}, \{\{1, 2\}, \{1\}\}, \{\{1, 2, 3\}, \{7, 9\}\}, \{\{a, b, c, d\}, \{c, d, x\}\}, \ldots \} \]

EXERCISES 3

A. 14 is 1 less than 15.
B. 37 is 4 more than 33.
C. 9 is 4 less than 13.
D. 7 is 2 more than 5.
E. 5 is 3 less than 8.

TASK 4

Given an ordered pair of whole numbers \((a, b)\), construct an interpretation representing its "more-less" property.

RULE 4

Select any element from the equivalence class (Rule 3) defined by its more-less property (Rule 2).

EXAMPLES 4

A. Given: \((2, 5)\),
   Answer:
   \[ \{\{a\}, \{a, b, c, d\}\} \].

B. Given: \((5, 4)\),
   Answer:
   \[ \{\{1, 2, 3, 4, 5, 6, 7\}, \{1, 2, 3, 4, 5, 6\}\} \].

EXERCISES 4

A. \((15, 12)\)
B. \((3, 9)\)
C. \((11, 5)\)
D. \((14, 23)\)
E. \((8, 6)\)
TASK 5

Given a task used to determine conservation of inclusion of one set (A) in another set (B), (e.g., a set of wooden objects (B) most of which are blue (A), and the rest white (C = B - A)), indicate how a child is apt to respond (1) if he (she) is a conserver, (2) if he (she) is not a conserver.

RULE 5

(1) If the child is a conserver then he is apt to respond that there are more objects having property B (e.g., wooden objects).

(2) If the child is not a conserver, then he is apt to respond that there are more objects having property A (e.g., blue wooden objects).

EXAMPLES 5

A. Given: A child is shown a set of 10 square objects, 8 of them are red and 2 of them are blue. The child is asked, "Are there more red objects or more square objects?"

Answer:

(1) The conserver will respond that there are more square objects.

(2) The non conserver will respond that there are more red objects (or that he does not know).

B. Given: A child is shown a set of 15 marbles, 10 of them are blue and 5 of them are red. The child is asked, "Are there more blue objects or more marbles?"

Answer:

(1) The conserver will respond that there are more marbles.

(2) The non conserver will respond that there are more blue objects (or that he does not know).

EXERCISES 5

A. A child is shown 8 cycles in a store, 6 of them are two-wheelers and 2 of them are three-wheelers. The child is asked, "Are there more cycles or are there more two-wheelers?"

B. A child is shown 9 lollypops, 6 of them are yellow and 3 of them are purple. The child is asked, "Are there more lollypops or are there more yellow objects?"

C. A child is shown 12 large balls, 8 of them are green and 4 of them are white. The child is asked, "Are there more green balls or large balls?"

D. A child is shown 7 triangular shapes, 5 of them are colored green and 2 of them are colored orange. The child is asked, "Are there more green objects or are
there more triangles?"

E. A child is shown 11 dolls, 8 of them are girl dolls and 3 of them are boy dolls. The child is asked, "Are there more dolls or girl dolls?"
SECTION 2. **Integers**

**TASK 6**

Given two ordered pairs of whole numbers \((a, b)\) and \((c, d)\), determine whether or not they are more-less equivalent.

**RULE 6**

Add the first element of the first pair to the second element of the second pair (i.e., \(a + d\)), and the second element of the first pair to the first element of the second pair (i.e., \(b + c\)). If the two sums are equal (i.e., if \(a + d = b + c\)), then \((a, b)\) and \((c, d)\) are more-less equivalent; otherwise, they are not.

**EXAMPLES 6**

A. Given: \((3, 7)\) and \((6, 10)\).

Answer:

\[3 + 10 = 13, \quad 7 + 6 = 13.\] Therefore \((3, 7)\) and \((6, 10)\) are more-less equivalent.

B. Given: \((1, 2)\) and \((15, 14)\).

Answer:

\[1 + 14 = 15, \quad 2 + 15 = 17.\] Therefore \((1, 2)\) and \((15, 14)\) are not more-less equivalent.

**EXERCISES 6**

A. \((6, 3)\) and \((8, 5)\)
B. \((2, 3)\) and \((7, 8)\)
C. \((5, 3)\) and \((4, 6)\)
D. \((12, 5)\) and \((8, 1)\)
E. \((3, 15)\) and \((20, 32)\)

**TASK 7**

Given two ordered pairs \((a, b)\) and \((c, d)\) determine whether or not they are more-less equivalent by finding the magnitude and direction of the difference represented by each pair.
RULE 7

Find the difference of the elements of each pair (larger-smaller). If the differences are equal and both first elements are less than (or both greater than) their corresponding second elements, then \((a, b)\) and \((c, d)\) are more-less equivalent; otherwise, they are not.

EXAMPLES 7

A. Given: \((7, 2)\) and \((13, 8)\).

Answer:

\[7 - 2 = 5, \quad 13 - 8 = 5.\] Both first elements are greater than the corresponding second elements, therefore \((7, 2)\) and \((13, 8)\) are more-less equivalent.

B. Given: \((3, 5)\) and \((9, 7)\).

Answer:

\[5 - 3 = 2, \quad 9 - 7 = 2.\] The first elements are neither both greater than nor both less than their corresponding second elements, therefore \((3, 5)\) and \((9, 7)\) are not more-less equivalent.

EXERCISES 7

A. \((6, 5)\) and \((3, 4)\)
B. \((15, 18)\) and \((3, 6)\)
C. \((17, 12)\) and \((8, 5)\)
D. \((11, 7)\) and \((6, 2)\)
E. \((4, 9)\) and \((8, 3)\)
SECTION 3. Representing Integers

TASK 8

Given an integer "n-lessness (moreness)," represent it as an equivalence class of ordered pairs of whole numbers.

RULE 8

Construct a set of ordered pairs of whole numbers in which the first number of each ordered pair is n-less (or n-more) than the second number.

EXAMPLES 8

A. Given: "5-moreness."

Construct:

\{(5, 0), (6, 1), (7, 2), (8, 3), ... \}.

B. Given: "7-lessness."

Construct:

\{(0, 7), (1, 8), (2, 9), (3, 10), ... \}.

EXERCISES 8

A. "3-lessness"
B. "6-moreness"
C. "0-lessness"
D. "2-moreness"
E. "8-lessness"

TASK 9

Given an integer "n-lessness (moreness)," represent it in set builder notation.

RULE 9

Write \{(x, x + n) \mid x \text{ is a whole number}\} for "n-lessness" and \{(x + n, x) \mid x \text{ is a whole number}\} for "n-moreness."

EXAMPLES 9

A. Given: 4-lessness.
Answer:

\[(x, x + 4) \mid x \text{ is a whole number}\].

B. Given: 2-moreness.

Answer:

\[(x + 2, x) \mid x \text{ is a whole number}\].

### EXERCISES 9

A. 3-lessness
B. 5-moreness
C. 1-lessness
D. 0-moreness
E. 8-lessness

### TASK 10

Given an integer "n-lessness (moreness)", represent it as a canonical pair where one of the elements in the pair is zero.

### RULE 10

Represent n-lessness by \((0, n)\) and n-moreness by \((n, 0)\).

### EXAMPLES 10

A. Given: 3-moreness,
   Answer: \((3, 0)\)

B. Given: 8-lessness,
   Answer: \((0, 8)\)

### EXERCISES 10

A. 7-lessness
B. 5-moreness
C. 0-lessness
D. 8-moreness
E. 3-lessness

### TASK 11

Given an integer "n-lessness (moreness)", represent it as a signed numeral.

### RULE 11

Represent n-lessness by \(-n\), n-moreness by \(+n\) and 0-moreness (lessness) by \(+0\).
EXAMPLES 11

A. Given: 2-lessness.
Answer:

\[ ^2 \]

B. Given: 5-moreness.
Answer:

\[ ^5 \]

EXERCISES 11

A. 5-lessness
B. 0-moreness
C. 14-moreness
D. 7-lessness
E. 12-moreness

---

TASK 12

Given an integer \( \pm n \) (\( \mp n \)), represent it as a point on the number line.

RULE 12

Draw a horizontal line, mark off a point and label it \( ^\pm 0 \). Mark off segments of equal length along the line in both directions from the point labeled \( ^\pm 0 \). Label the point \( n \) units to the right (left) of \( ^\pm 0 \) as \( \pm n \) (\( \mp n \)).

EXAMPLES 12

A. Given: \( \pm 3 \).
Answer:

\[ \begin{array}{c}
0 \\
\pm 3
\end{array} \]

B. Given: \( \mp 4 \).
Answer:

\[ \begin{array}{c}
-4 \\
\pm 0
\end{array} \]

EXERCISES 12

A. \( \pm 1 \)
B. \( \mp 5 \)
C. \( \pm 4 \)
D. \( \mp 3 \)
E. \( \pm 6 \)
TASK 13

Given a whole number \( n \), represent it as a point on the number line.

RULE 13

Represent \( n \) in the same way as \( t \) in Rule 12.

EXAMPLES 13

A. Given: 4.
Answer: 

B. Given: 7.
Answer: 

EXERCISES 13

A. 2  
B. 13  
C. 8  
D. 1  
E. 6

TASK 14

Given a non-negative rational number representative, \( \frac{a}{b} \), represent it as a point on the number line.

RULE 14

Draw a number line as in Rule 12.
(a) If \( \frac{a}{b} = 0 \), represent \( \frac{a}{b} \) by the point labeled 0.
(b) If \( \frac{a}{b} < 1 \), partition the interval between 0 and 1 into \( b \) equal sized pieces, count (starting at 0) \( a \) of them, and label that point \( \frac{a}{b} \).
(c) If \( \frac{a}{b} > 1 \), write \( \frac{a}{b} \) as a mixed number by dividing \( b \) into \( a \) and expressing the result as \( q \frac{r}{b} \). Partition the interval between \( q \) and \( q + 1 \) into \( b \) equal sized pieces, count (starting at \( q \)) \( r \) of them, and label the end point of the last piece \( \frac{a}{b} \).
EXAMPLES 14

A. Given: \( \frac{3}{4} \).
Answer:
Apply part (b):

\[
\begin{array}{cccccc}
-1 & 0 & \frac{3}{4} & 1 & 2 \\
\end{array}
\]

B. Given: \( \frac{17}{5} \).
Answer:
Apply part (c), \( \frac{17}{5} = \frac{32}{5} \).

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & \frac{17}{5} & 4 \\
\end{array}
\]

EXERCISES 14

A. \( \frac{4}{6} \)
B. \( \frac{13}{2} \)
C. \( \frac{0}{3} \)
D. \( \frac{3}{8} \)
E. \( \frac{10}{3} \)
SECTION 4. Addition

TASK 15

Given two integers \( x \) and \( y \), find the sum \( x + y \).

RULE 15

Choose representatives \( (a, b) \) of \( x \) and \( (a', b') \) of \( y \). Add the first elements \( (a + a') \) and the second elements \( (b + b') \). The sum is represented by the ordered pair \( (a + a', b + b') \). Find the difference \( (d) \) of \( a + a' \) and \( b + b' \) (larger-smaller). If \( a + a' \) is greater than \( b + b' \), then \( x + y = +d \). If \( a + a' \) is less than \( b + b' \), then \( x + y = -d \).

EXAMPLES 15

A. Given: \( +5 \) and \( -3 \).
   Answer:
   Let \( (6, 1) \) represent \( +5 \) and \( (4, 7) \) represent \( -3 \). The sum is represented by \( (10, 8) \). \( 10 - 8 = 2 \). Therefore \( +5 + -3 = +2 \).

B. Given: \( -3 \) and \( -4 \).
   Answer:
   Let \( (1, 4) \) represent \( -3 \) and \( (5, 9) \) represent \( -4 \). The sum is represented by \( (6, 13) \). \( 13 - 6 = 7 \). Therefore \( -3 + -4 = -7 \).

EXERCISES 15

A. \( -2, -5 \)
B. \( -2, -4 \)
C. \( +1, +3 \)
D. \( -4, +5 \)
E. \( -1, -3 \)

\( \sqrt{ \text{TASK 16} } \)

Given two integers \( x \) and \( y \), determine whether or not the closure property for addition of integers holds for \( x \) and \( y \).

RULE 16

Apply Rule 19, Chapter 5 to Rule 11, Chapter 5 and apply the derived rule to \( x \) and \( y \).
Chapter 8
Section 4

EXERCISES 16

A. $-7, -9$
B. $+8, -12$
C. $+4, +9$
D. $-3, -8$
E. $-9, +14$


TASK 17

Given two integers $x$ and $y$, show that the operation of addition of integers is well-defined for $x$ and $y$.

RULE 17

Apply Rule 23, Chapter 5 to $x$ and $y$.

EXERCISES 17

A. $-7$ and $+3$
B. $+1$ and $+9$
C. $-5$ and $-3$
D. $+8$ and $-7$
E. $+3$ and $+4$


TASK 18

Given two integers $x$ and $y$, find the sum by using the four-case algorithm.

RULE 18

1. If both signs are $+$, add the magnitudes (as if they were whole numbers) and affix "$+$" to the result.
2. If both signs are $-$, add the magnitudes and affix "$-$" to the result.
3. If the sign of the integer with the larger magnitude is $+$ and that of the smaller magnitude is $-$, then subtract the smaller magnitude and affix "$+$" to the result.
4. If the sign of the integer with the larger magnitude is $-$ and that of the smaller magnitude is $+$, then subtract the smaller magnitude and affix "$-$" to the result.

* If both integers are of equal magnitude, either may be picked as the smaller (larger) in order to apply the rule.

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EXAMPLES 18

A. Given: +7 and −3.

Answer:

Apply case (3). 7 − 3 = 4. +7 + −3 = +4.

B. Given: −5 and −6.

Answer:

Apply case (2). 5 + 6 = 11. −5 + −6 = −11.

EXERCISES 18

A. +4, −13
B. −7, −9
C. +16, −5
D. +11, +13
E. +8, −17

TASK 19

Given two integers $x$ and $y$, represent the operation of addition on a number line.

RULE 19

Represent $x$ on the number line (Rule 12). Starting at the point corresponding to $x$ mark off $y$ units to the right of $x$ if the sign of $y$ is $+$; otherwise mark off $y$ units to the left of $x$. The terminal point corresponds to the sum.

EXAMPLES 19

A. Given: +7 and −4.

Construct:

The sum is +3.
B. Given: \(-3\) and \(-2\).

Construct:

\[ \begin{array}{cccccccccc}
-5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\end{array} \]

The sum is \(-5\).

EXERCISES 19

A. \(+5\) and \(-7\)
B. \(-2\) and \(-4\)
C. \(-6\) and \(+5\)
D. \(+2\) and \(+4\)
E. \(-8\) and \(+6\)

TASK 20

Given two integers \(x\) and \(y\), show that the definition of addition and the four-case algorithm yield the same result for the sum \(x + y\).

RULE 20

Apply Rule 15 to \(x\) and \(y\). Next apply Rule 18 to \(x\) and \(y\). (Observe that both give the same result.)

EXAMPLES 20

A. Given: \(+7\) and \(-5\).

Answer:

Applying the definition of addition (Rule 15), choose representatives \((8, 1)\) and \((2, 7)\) of \(+7\) and \(-5\) respectively. Adding corresponding elements gives \((10, 8)\) which represents the integer \(-2\). Applying the four-case algorithm (Rule 18) gives the sum \(-2\).

B. Given: \(-3\) and \(-6\).

Answer:

Applying the definition of addition (Rule 15), choose representatives \((4, 7)\) and \((6, 12)\) of \(-3\) and \(-6\) respectively. Adding corresponding elements gives
(10, 19) which represents the integer $-9$. Applying the four-case algorithm (Rule 18) gives the sum $-9$.

EXERCISES 20

A. $+5$ and $+7$
B. $-3$ and $+4$
C. $+6$ and $-4$
D. $-7$ and $-18$
E. $+6$ and $-11$

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TASK 21

Given a problem situation involving magnitudes and directions solve the problem using the four-case algorithm.

RULE 21

Assign $+$ to one direction and $-$ to the other. Prefix each magnitude with the appropriate sign and then apply Rule 18.

EXAMPLES 21

A. Mrs. Smith told her class that the temperature rose 3 degrees between 9 and 10 o'clock and fell 7 degrees between 10 and 11 o'clock. What was the change in temperature between 9 and 11 o'clock?

Answer:

Assign $+$ to rise and $-$ to fall in temperature.

$+3 + (-7) = -4$. The temperature fell 4 degrees.

B. During the game of Giant Steps, Jim was told to take 2 steps back and then 5 steps forward. What was his final position?

Answer:

Assign $+$ to forward and $-$ to backward.

$-2 + +5 = +3$. Jim took 3 steps forward.
EXERCISES 21

A. Jack received $10 payment for cutting lawns and promptly paid his friend Bill the $6 he owed him. What was Jack's net gain (or loss) for the day?

B. Tom estimated that he used 8 gallons of gas on a recent shopping trip. Since he had borrowed his mother's car, he put 13 gallons of gas into the car before returning it. How many gallons more (or less) did the car have after Tom returned it?

C. During the morning Mike lost 12 marbles to Jim and in the afternoon won 7 marbles from Frank. What was his total gain (or loss) for the day?

D. Early in the afternoon Mrs. Brown noticed that the temperature in the house had risen 10 degrees since morning. She put the air conditioning on and shortly thereafter she noticed that the temperature had gone down 7 degrees. What was the total rise (or fall) in temperature?

E. The pilot informed his passengers that after take-off he had climbed to an altitude of 20,000 feet and then one hour later had dropped down 3,000 feet. At what altitude were they flying?
SECTION 5. Subtraction

TASK 22

Given two integers \( x \) and \( y \), find the difference \( x - y \).

RULE 22

Choose representatives \((a, b)\) of \( x \) and \((a', b')\) of \( y \) in such a way that \( a \geq a' \) and \( b \geq b' \). Write the ordered pair whose first element is \( a - a' \) and whose second element is \( b - b' \) (i.e., \((a - a', b - b')\)). Find the difference \( d \) between \( a - a' \) and \( b - b' \) (larger - smaller). If \( a - a' \) is greater than \( b - b' \), then \( x - y = a \). If \( a - a' \) is less than \( b - b' \), then \( x - y = b \).

EXAMPLES 22

A. Given: \(+3\), \(-5\). Find \(+3 - (-5)\).

Answer:

Let \((10, 7)\) represent \(+3\) and \((1, 6)\) represent \(-5\). The difference is represented by \((9, 1)\). \(9 - 1 = 8\). Therefore \(+3 - (-5) = 8\).

B. Given: \(-6\), \(-2\). Find \(-6 - (-2)\).

Answer:

Let \((8, 14)\) represent \(-6\) and \((4, 6)\) represent \(-2\). The difference is represented by \((4, 8)\). \(8 - 4 = 4\). Therefore \(-6 - (-2) = 4\).

EXERCISES 22

A. \(+7 - (-2)\)
B. \(-5 - (-9)\)
C. \(+4 - (-6)\)
D. \(-9 - (-3)\)
E. \(+6 - (-2)\)

\(\sqrt{\text{TASK 23}}\)

Given an addition problem \( \Box + y = x \), show that subtraction may be viewed as the inverse of addition by solving the corresponding subtraction problem.

RULE 23

Apply Rule 35, Chapter 5 to \( \Box + y = x \).
EXERCISES 23

A. \( \Box + 3 = 7 \)
B. \( \Box + 8 = 5 \)
C. \( \Box + 9 = 4 \)
D. \( \Box + 6 = 12 \)
E. \( \Box + 2 = 6 \)

\[ \text{\sqrt{\text{TASK 24}}} \]

Given two integers \( x \) and \( y \), determine whether or not the closure law for subtraction of integers holds for \( x \) and \( y \).

RULE 24

Apply Rule 19, Chapter 5 to Rule 11, Chapter 5 and apply the derived rule to \( x \) and \( y \).

EXERCISES 24

A. \( +6 \) and \( -7 \)
B. \( -4 \) and \( -8 \)
C. \( -9 \) and \( +12 \)
D. \( +5 \) and \( +7 \)
E. \( -8 \) and \( +3 \)

\[ \text{\sqrt{\text{TASK 25}}} \]

Given two integers \( x \) and \( y \), show that the operation of subtraction of integers is well-defined for \( x \) and \( y \).

RULE 25

Apply Rule 23, Chapter 5 to \( x \) and \( y \).

EXERCISES 25

A. \( +6 \) and \( +8 \)
B. \( +4 \) and \( +3 \)
C. \( +2 \) and \( -5 \)
D. \( -3 \) and \( -7 \)
E. \( +4 \) and \( -1 \)
TASK 26

Given two integers \( x \) and \( y \), find the difference using the four-case algorithm.

RULE 26

Change the sign of the integer being subtracted and then add using Rule 18.

EXAMPLES 26

A. Given: \(-6, +3\). Find \(-6 - +3\).

Answer:

Changing the sign of \( +3 \) and adding (Rule 18) obtain \(-6 - +3 = -6 + -3 = -9\).

B. Given: \(+4, +7\). Find \(+4 - +7\).

Answer:

Changing the sign of \( +7 \) and adding (Rule 18) obtain \(+4 - +7 = +4 + -7 = -3\).

EXERCISES 26

A. \(-5 - -3\)
B. \(+6 - -4\)
C. \(+3 - -3\)
D. \(-7 - +4\)
E. \(-2 - -6\)

TASK 27

Given two integers \( x \) and \( y \), represent the operation of subtraction on a number line.

RULE 27

Represent \( x \) on the number line (Rule 12). Starting at the point corresponding to \( x \) undo \( y \) (i.e., mark off \( y \) units to the left of \( x \) if the sign of \( y \) is \( + \); otherwise mark off \( y \) units to the right of \( x \)). The terminal point corresponds to the difference.

EXAMPLES 27

A. Given: \(-6, -8\). Find \(-6 - -8\).
Answer:

Represent $-6$ on the number line. Undo $-8$ (i.e., mark off 8 units to the right of $-6$).

Hence, $-6 - (-8) = +2$

B. Given: $-4$, $-6$. Find $-4 - (-6)$.

Answer:

Represent $-4$ on the number line. Undo $-6$ (i.e., mark off 6 units to the left of $-4$).

Hence, $-4 - (-6) = +2$.

EXERCISES 27

A. $-3 - (+4)
B. $+5 - (-2)
C. $+2 - (+6)
D. $-4 - (+7)
E. $+6 - (-3)$

TASK 28

Given two integers $x$ and $y$, show that the definition of subtraction and the four-case algorithm for subtraction yield the same result.

RULE 28

Apply Rule 22 to $x$ and $y$. Next apply Rule 26 to $x$ and $y$. (Observe that both give the same result.)
EXAMPLES 28

A. Given: $-7$ and $+3$.

Answer:

Applying the definition of subtraction (Rule 22), choose representatives $(8, 15)$ and $(4, 1)$ of $-7$ and $+3$ respectively. Subtracting corresponding elements gives $(4, 14)$ which represents the integer $-10$. Applying the four-case algorithm (Rule 26) gives the difference $-10$.

B. Given: $+6$ and $+3$.

Answer:

Applying the definition of subtraction (Rule 22), choose representatives $(14, 8)$ and $(7, 4)$ of $+6$ and $+3$ respectively. Subtracting corresponding elements gives $(7, 4)$ which represents the integer $+3$. Applying the four-case algorithm (Rule 26) gives the difference $+3$.

EXERCISES 28

A. $-5$ and $-4$
B. $-6$ and $-8$
C. $+3$ and $+4$
D. $+2$ and $+7$
E. $-7$ and $+3$
SECTION 6. Multiplication

TASK 29

Given two integers \( x \) and \( y \), find the product \( x \times y \).

RULE 29

Choose representatives \( (a, b) \) of \( x \) and \( (a', b') \) of \( y \). Write the ordered pair \( ((a \times a') + (b \times b'), (a \times b') + (b \times a')) \). This ordered pair represents the product \( x \times y \).

EXAMPLES 29

A. Given: \( +3 \times -2 \).
   Answer:
   Choose representatives \( (6, 3) \) and \( (2, 4) \) of \( +3 \) and \( -2 \) respectively. The product is represented by \( ((6 \times 2) + (3 \times 4), (6 \times 4) + (3 \times 2)) \) = \( (24, 30) \). Therefore, \( +3 \times -2 = -6 \).

B. Given: \( -4 \times -5 \).
   Answer:
   Choose representatives \( (2, 6) \) and \( (4, 9) \) of \( -4 \) and \( -5 \) respectively. The product is represented by \( ((2 \times 4) + (6 \times 9), (2 \times 9) + (6 \times 4)) \) = \( (62, 42) \). Therefore, \( -4 \times -5 = +20 \).

EXERCISES 29

A. \( -7 \times +4 \)
B. \( -3 \times -2 \)
C. \( +4 \times +6 \)
D. \( +5 \times -3 \)
E. \( -8 \times -7 \)

\( \sqrt{\text{TASK 30}} \)

Given two integers \( x \) and \( y \), determine whether or not the closure law for multiplication of integers holds for \( x \) and \( y \).
RULE 30

Apply Rule 19, Chapter 5 to Rule 11, Chapter 5 and apply the derived rule to \( x \) and \( y \).

EXERCISES 30

A. \(-4\) and \(+6\)
B. \(+3\) and \(+5\)
C. \(-2\) and \(-7\)
D. \(-8\) and \(-2\)
E. \(-6\) and \(+5\)

\[ \text{\ldots} \]

\( \sqrt{\text{TASK 31}} \)

Given two integers \( x \) and \( y \), show that the operation of multiplication of integers is well-defined for \( x \) and \( y \).

RULE 31

Apply Rule 23, Chapter 5 to \( x \) and \( y \).

EXERCISES 31

A. \(+3\) and \(-6\)
B. \(-2\) and \(+5\)
C. \(+7\) and \(+4\)
D. \(+2\) and \(-8\)
E. \(+4\) and \(-3\)

\[ \text{\ldots} \]

TASK 32

Given two integers \( x \) and \( y \), show that the product may be viewed in terms of Cartesian products.

RULE 32

Construct a pair of disjoint sets \((A, B)\) representing \( x \) and a pair of disjoint sets \((C, D)\) representing \( y \). Form the Cartesian products of (1) the like pairs, \( A \times C \) and \( B \times D \), (2) the unlike pairs, \( A \times D \) and \( B \times C \). Let \( s \) be the number of like pairs and \( t \) the number of unlike pairs. Then, find the integer represented by the pair \((s, t)\). Next apply Rule 29 to \( x \) and \( y \). Finally, note that the last two
results are the same.

EXAMPLES 32

A. Given: $3 \times -2$.

Answer:

$\{a\}, \{1, 2, 4, 5\}$ represents $3$ and $\{c, k\}, \{3, 7, 8, 9\}$ represents $-2$.

$\{a\} \times \{c, k\} = \{(a, c), (a, k)\}$.

$\{1, 2, 4, 5\} \times \{3, 7, 8, 9\} = \{(1, 3), (1, 7), (1, 8), (1, 9), (2, 3), (2, 7), (2, 8), (2, 9), (4, 3), (4, 7), (4, 8), (4, 9), (5, 3), (5, 7), (5, 8), (5, 9)\}$.

The total number of like pairs is 18.

$\{a\} \times \{3, 7, 8, 9\} = \{(a, 3), (a, 7), (a, 8), (a, 9)\}$.

$\{1, 2, 4, 5\} \times \{c, k\} = \{(1, c), (1, k), (2, c), (2, k), (4, c), (4, k), (5, c), (5, k)\}$.

The total number of unlike pairs is 12. The product is represented by $(18, 12)$, i.e., $^6$. The integer $^6$ is also obtained by applying Rule 29.

B. Given: $+1$ and $-2$.

Answer:

$\{2, 4\}, \{a\}$ represents $+1$ and $\{5\}, \{h, m, t\}$ represents $-2$.

$\{2, 4\} \times \{5\} = \{(2, 5), (4, 5)\}$. $\{a\} \times \{h, m, t\} = \{(a, h), (a, m), (a, t)\}$.

The total number of like pairs is 5.

$\{2, 4\} \times \{h, m, t\} = \{(2, h), (2, m), (2, t), (4, h), (4, m), (4, t)\}$. $\{a\} \times \{5\} = \{(a, 5)\}$.

The total number of unlike pairs is 7. The product is represented by $(5, 7)$, i.e., $^-2$. The integer $^-2$ is also obtained by applying Rule 29.

EXERCISES 32

A. $-1$ and $+1$
B. $+3$ and $-4$
C. $-3$ and $-1$
D. $+5$ and $-2$
E. $-4$ and $-2$
TASK 33

Given two integers $x$ and $y$, find the product using the four-case algorithm.

RULE 33

(a) If both signs are $+$, multiply the magnitudes (as if they were whole numbers) and affix "$+$" to the result.
(b) If the sign of $x$ is $-$ and the sign of $y$ is $+$, multiply the magnitudes and affix "-$" to the result.
(c) If the sign of $x$ is $+$ and the sign of $y$ is $-$, multiply the magnitudes and affix "-$" to the result.
(d) If both signs are $-$, multiply the magnitudes and affix "$+$" to the result.

EXAMPLES 33

A. Given: $-7$ and $+6$.

Answer:

Apply part (b): $7 \times 6 = 42$.

$-7 \times +6 = -42$.

B. Given: $-5$ and $-3$.

Answer:

Apply part (d): $5 \times 3 = 15$.

$-5 \times -3 = +15$.

EXERCISES 33

A. $-4$, $-3$
B. $+2$, $-9$
C. $+5$, $+7$
D. $+6$, $+8$
E. $-9$, $-6$

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TASK 34

Given two integers $x$ and $y$, represent the operation of multiplication on a number line.

RULE 34

(Use (a), (b), (c), and (d) as in Rule 33)
EXAMPLES 34

A. Given: $^+2$ and $^-6$.

Answer:

Apply part (b)
The product is $-12$.

B. Given: $-3$ and $-5$.

Answer:

Apply part (d)

The product is $+15$.

**EXERCISES 34**

A. $+6$ and $-3$
B. $-4$ and $-2$
C. $-8$ and $+3$
D. $+2$ and $+6$
E. $+5$ and $-2$

**TASK 35**

Given two integers $x$ and $y$, show that the definition of multiplication and the four-case algorithm yield the same result for the product $x \times y$.

**RULE 35**

Apply Rule 29 to $x$ and $y$. Next apply Rule 33 to $x$ and $y$. (Observe that both give the same result.)
EXAMPLES 35

A. Given: \( +3 \) and \( -4 \).

Answer:

Applying the definition of multiplication (Rule 29), choose representatives \((5, 2)\) and \((4, 8)\) of \( +3 \) and \( -4 \) respectively. The product is represented by \(((5 \times 4) + (2 \times 8), (5 \times 8) + (2 \times 4)) = (36, 48)\). Therefore \( +3 \times -4 = -12 \).

Applying the four-case algorithm (Rule 33) gives the product \(-12\).

B. Given: \( -5 \) and \( -1 \).

Answer:

Applying the definition of multiplication (Rule 29), choose representatives \((1, 6)\) and \((3, 4)\) of \( -5 \) and \( -1 \) respectively. The product is represented by \(((1 \times 3) + (6 \times 4), (1 \times 4) + (6 \times 3)) = (27, 22)\). Therefore \( -5 \times -1 = +5 \).

Applying the four-case algorithm (Rule 33) gives the product \(+5\).

EXERCISES 35

A. \(-7, +5\)
B. \(+3, +4\)
C. \(-6, -5\)
D. \(-8, -2\)
E. \(-4, +9\)

TASK 36

Given two integers \( x \) and \( y \), describe a real-world interpretation of the product \( x \times y \).

RULE 36

Describe a situation where \( x \) represents a directed state or operator (e.g., water flowing in or out of a tank, debts and credits) and \( y \) represents a higher order directed operator which acts on \( x \) (e.g., number of times an operator is repeated, time in the future and time past). The product \( x \times y \) is represented by the resulting directed state or operator (e.g., debt or credit, amount of liquid in a tank).

EXAMPLES 36

A. Given: \( -4 \) and \( +3 \).
Answer:

Water is pouring out of a tank at the rate of 4 gallons per minute. There
will be 12 fewer gallons \((-4 \times +3 = -12)\) in the tank 3 minutes from now.

B. Given: \(-3\) and \(+5\).

Answer:

Mrs. Jones went shopping for shrubs. Azaleas were on sale for $3 each, so
Mrs. Jones purchased 5 of them. She came home with \(-3 \times +5 = -15\) dollars, i.e.,
15 dollars less than when she left home.

EXERCISES 36

A. \(+2, +5\)
B. \(+6, -3\)
C. \(-5, -6\)
D. \(-7, +3\)
E. \(-4, -6\)
SECTION 7. Division

\( \sqrt{\text{TASK 37}} \)

Given a multiplication problem \( \square \times y = x \), show that division may be viewed as the inverse of multiplication by solving the corresponding division problem.

RULE 37

Apply \( \mathcal{R} \)ule 35, Chapter 5 to \( \square \times y = x \).

EXERCISES 37

A. \( \square \times 2 = 6 \)
B. \( \square \times 4 = 32 \)
C. \( \square \times 7 = 14 \)
D. \( \square \times 3 = 24 \)
E. \( \square \times 5 = 38 \)

\( \text{___} \cdot \text{___} \)

\( \text{TASK 38} \)

Given two integers \( x \) and \( y \) (where the magnitude of \( y \) is a divisor of the magnitude of \( x \)), find the quotient \( x \div y \) using the four-case algorithm.

RULE 38

Apply Rule 33 where "multiply" is replaced by "divide" and "result" by "quotient."

EXAMPLES 38

A. Given: \( -12 \) and \( -3 \).

Answer:

Apply part (b). \( -12 \div -3 = 4 \)
\( -12 \div -3 = 4 \)

B. Given: \( -42 \) and \( -7 \).

Answer:

Apply part (d). \( -42 \div -7 = 6 \)
\( -42 \div -7 = 6 \)
EXERCISES 38

A. $4$, $-7$
B. $-9$, $-3$
C. $-16$, $-4$
D. $+26$, $+13$
E. $-15$, $+5$

TASK 39

Given two integers $x$ and $y$ (where the magnitude of $y$ is a divisor of the magnitude of $x$), represent the operation of division on a number line.

RULE 39

(a) If both signs are $+$ divide the magnitudes as if they were natural numbers and affix "$+$" to the result.

For $+m + +n$:

\[
\text{take away } +n \text{ from } +m, \ m + n \text{ times}
\]

(b) If the sign of $x$ is $-$ and the sign of $y$ is $+$, divide the magnitudes and affix "$-$" to the result.

For $-m + +n$:

\[
\text{undo take away (add) } +n \text{ to } -m, \ m + n \text{ times}
\]

(c) If the sign of $x$ is $+$ and $y$ $-$, divide the magnitudes and affix "$-$" to the result.

For $+m + -n$:
undo take away (add) \(-n\) to \(+m\), \(m + n\) times

(d) If both signs are -, divide the magnitudes and affix "+" to the result.

For \(-m + -n\)

Examples 39

A. Given: \(-4\) and \(-12\).

Construct: (using (d))

\[-12 \quad -4 \quad 0\]

\[-12 + -4 = +3.\]

B. Given: \(+4\) and \(-2\).

Construct: (using (c))

\[+4 \quad -2 \quad 0 \quad +4\]

"undoing" 2 jumps of \(-2\) gives \(+4 + -2 = -2\).
EXERCISES 29

A. $-14$ and $+7$
B. $-9$ and $+3$
C. $-10$ and $-5$
D. $+16$ and $-8$
E. $-14$ and $-7$

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TASK 40

Given two integers, $x$ and $y$, (where the magnitude of $y$ is a divisor of the magnitude of $x$), show that the quotient obtained by applying the four case algorithm for division to $x$ and $y$, satisfies the equation $\Box \times y = x$.

RULE 40

Apply Rule 38 to $x$ and $y$ and then substitute the resulting quotient (answer) in $\Box \times y = x$. (Notice that the product $\Box \times y$ equals $x$.)

EXAMPLES 40

A. Given: $+8$ and $-4$.
   Answer:
   Apply Rule 38 (part c) to obtain $+8 \div -4 = -2$. Substituting $-2$ in $\Box \times -4 = +8$. Notice that $-2 \times -4$ equals $+8$.

   Answer:
   Apply Rule 38 (part d) to obtain $-24 \div -8 = +3$. Substituting $+3$ in $\Box \times -8 = -24$. Notice that $+3 \times -8$ equals $-24$.

EXERCISES 40

A. $-36$ and $+18$
B. $+18$ and $+9$
C. $+14$ and $-7$
D. $-38$ and $-2$
E. $-44$ and $+11$

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Section 8. Properties of Systems of Integers

TASK 41

Given a quadratic equation \( ax^2 + bx + c = 0 \), determine whether or not a given integer \( z \) satisfies the equation.

RULE 41

Replace each occurrence of \( x \) by \( z \) in the given equation. Perform the indicated operations by applying Rules 33 and 18. If the result is 0, then \( z \) satisfies the equation. Otherwise, it does not.

EXAMPLES 41

A. Determine whether \( -5 \) satisfies \( +2x^2 + +9x + +5 = 0 \).

Answer:

\[
+2 x (-5)^2 + +9 (-5) + -5 = +2 \times +25 + -45 + -5 \\
= +50 + -45 + -5 \\
= +5 + -5 \\
= +0
\]

Therefore, \( -5 \) satisfies the given equation.

B. Determine whether \( +3 \) satisfies \( +2x^2 + -2x + -24 = 0 \).

Answer:

\[
+2 x (+3)^2 + (-2 x +3) + -24 = (+2 x +9) + -6 + -24 \\
= +18 + -6 + -24 \\
= +12 + -24 \\
= -12
\]

Therefore, \( +3 \) does not satisfy the given equation.

EXERCISES 41

Determine whether or not the given integer satisfies the given equation.

A. \( +4x^2 + -4x + +8 = 0 \) \( z = +1 \)
B. \( -x^2 + -2x + -1 = 0 \) \( z = +3 \)
C. \( x^2 + +3x + -10 = 0 \) \( z = -5 \)
D. \( -2x^2 + -13x + -7 = 0 \) \( z = +7 \)
E. \( -3x^2 + +5x + -11 = 0 \) \( z = -4 \)
8.1. Closure

**TASK 42**

Given two integers $x$ and $y$, determine whether or not the closure property for division of integers holds for $x$ and $y$.

**RULE 42**

Apply Rule 19, Chapter 5, to Rule 11, Chapter 5, and apply the derived rule to $x$ and $y$.

**EXERCISES 42**

A. $-13$, $+5$
B. $+16$, $-4$
C. $-18$, $-10$
D. $+17$, $-9$
E. $-11$, $+10$
8.2. **Associativity and Commutativity**

**TASK 43**

Given three integers \(x, y,\) and \(z,\) determine whether or not the associative property for addition of integers holds for \(x, y,\) and \(z,\)

**RULE 43**

Apply Rule 19, Chapter 5 to Rule 12, Chapter 5 and apply the derived rule to \(x, y,\) and \(z,\)

**EXERCISES 43**

A. \(-7, +6, -3\)
B. \(+5, +9, -2\)
C. \(-4, -3, +9\)
D. \(+8, -5, +7\)
E. \(-9, +3, -4\)

**TASK 44**

Given two integers \(x\) and \(y,\) determine whether or not the commutative property for addition of integers holds for \(x\) and \(y,\)

**RULE 44**

Apply Rule 19, Chapter 5, to Rule 13, Chapter 5, and apply the derived rule to \(x\) and \(y,\)

**EXERCISES 44**

A. \(-7, +6\)
B. \(+5, -9\)
C. \(-3, +8\)
D. \(+6, +4\)
E. \(-5, +9\)

**TASK 45**

Given three integers, \(x, y,\) and \(z,\) determine whether or not the associative property for multiplication of integers holds for \(x, y,\) and \(z,\)
RULE 45

Apply Rule 19, Chapter 5, to Rule 12, Chapter 5, and apply the derived rule to \( x, y, \) and \( z. \)

EXERCISES 45

A. \(-3, -5, +9\)
B. \(+7, -4, +8\)
C. \(+3, +5, -11\)
D. \(-9, +5, -7\)
E. \(+3, -8, -6\)

\( \sqrt{T} \)ask 46

Given two integers \( x \) and \( y, \) determine whether or not the commutative property for multiplication of integers holds for \( x \) and \( y. \)

RULE 46

Apply Rule 19, Chapter 5 to Rule 13, Chapter 5 and apply the derived rule to \( x \) and \( y. \)

EXERCISES 46

A. \(-9, +6\)
B. \(+7, +8\)
C. \(-3, -8\)
D. \(-5, -9\)
E. \(-7, +3\)

\( \sqrt{T} \)ask 47

Given three integers \( x, y, \) and \( z, \) determine whether or not the associative property for subtraction of integers holds for \( x, y, \) and \( z. \)

RULE 47

Apply Rule 19, Chapter 5 to Rule 12, Chapter 5 and apply the derived rule to \( x, y, \) and \( z. \)
EXERCISES 47
A. \(-7, +6, -4\)
B. \(+3, -9, +3\)
C. \(+6, -5, +3\)
D. \(+7, -9, +2\)
E. \(+3, -8, +7\)

\[\text{\_\_\_\_\_\_\_} \quad \text{\_\_\_\_\_\_\}_} \]

\sqrt{\text{\textsc{Task 48}}}

Given three integers \(x, y,\) and \(z,\) determine whether or not the associative property for division of integers holds for \(x, y,\) and \(z.\)

\textbf{Rule 48}

Apply Rule 19, Chapter 5 to Rule 12, Chapter 5 and apply the derived rule to \(x, y,\) and \(z.\)

EXERCISES 48
A. \(+9, -7, +6\)
B. \(-8, +4, -2\)
C. \(+12, +6, -3\)
D. \(-15, -5, +7\)
E. \(-18, -9, -3\)

\[\text{\_\_\_\_\_\_\_} \quad \text{\_\_\_\_\_\_\}_} \]

\sqrt{\text{\textsc{Task 49}}}

Given two integers \(x\) and \(y,\) determine whether or not the commutative property for subtraction of integers holds for \(x\) and \(y.\)

\textbf{Rule 49}

Apply Rule 19, Chapter 5 to Rule 13, Chapter 5 and apply the derived rule to \(x\) and \(y.\)

EXERCISES 49
A. \(+7, +6\)
B. \(+12, -9\)
C. \(-8, +3\)
D. \(+7, -4\)
E. \(-2, -6\)

\[\text{\_\_\_\_\_\_\_} \quad \text{\_\_\_\_\_\_\}_} \]
Task 50

Given two integers \( x \) and \( y \), determine whether or not the commutative property for division of integers holds for \( x \) and \( y \).

Rule 50

Apply Rule 19, Chapter 5 to Rule 13, Chapter 5, and apply the derived rule to \( x \) and \( y \).

Exercises 50

A. \(+9, -3\)
B. \(-5, -7\)
C. \(+8, +4\)
D. \(-12, +3\)
E. \(+6, -4\)
8.3. Identity

**VTask 51**

Given an integer \( x \), determine whether or not \( +0 = \{(0, 0), (1, 1), (2, 2), \ldots \} \) is an additive identity for \( x \).

**Rule 51**

Apply Rule 19, Chapter 5 to Rule 14, Chapter 5, and apply the derived rule to \( x \).

**Exercises 51**

A. \( +7 \)
B. \( -9 \)
C. \( +4 \)
D. \( -3 \)
E. \( +5 \)

(Note: \( +0 \) is the additive identity for all integers.)

**VTask 52**

Given an integer \( x \), determine whether or not \( +0 \) is a subtraction identity for \( x \).

**Rule 52**

Apply Rule 19, Chapter 5 to Rule 14, Chapter 5 and apply the derived rule to \( x \).

**Exercises 52**

A. \( +3 \)
B. \( -6 \)
C. \( +4 \)
D. \( -3 \)
E. \( +2 \)

(Note: Since \( x - +0 = x \) but \( +0 - x \neq x \), \( +0 \) is called a right identity for subtraction.)

**VTask 53**

Given an integer \( x \), determine whether or not \( +1 = \{(1,0), (2,1), (3,2), \ldots \} \) is a multiplicative identity for \( x \).
RULE 53

Apply Rule 19, Chapter 5 to Rule 14, Chapter 5 and apply the derived rule to $x$.

EXERCISES 53

A. $-7$
B. $+9$
C. $-3$
D. $+4$
E. $-6$
(Note: $+1$ is the multiplicative identity for all integers.)

\[ \text{________} \cdot \text{________} \]

\[ \sqrt{\text{TASK 54}} \]

Given an integer $x$, determine whether or not $+1$ is a division identity for $x$.

RULE 54

Apply Rule 19, Chapter 5 to Rule 14, Chapter 5 and apply the derived rule to $x$.

EXERCISES 54

A. $-8$
B. $+4$
C. $-9$
D. $+6$
E. $-3$
(Note: $+1$ is a \textbf{right} identity for division. (See Task 52).)

\[ \text{________} \cdot \text{________} \]
8.4. Inverse

**TASK 55**

Given an integer $x$, give its additive inverse and verify the inverse property.

**RULE 55**

Change the sign of $x$. This gives its additive inverse $x'$. Show that $x + x' = +0$ (Rule 21). (Note: $+0$ is the additive identity (Rule 51).)

**EXAMPLES 55**

A. Given: $-6$

Answer:

The additive inverse of $-6$ is $+6$. $-6 + +6 = +0$

B. Given: $-32$

Answer:

The additive inverse of $-32$ is $+32$. $+32 + -32 = +0$

**EXERCISES 55**

A. $+9$
B. $-4$
C. $+15$
D. $-2$
E. $+8$

**\sqrt{TASK 56}**

Given two integers $x$ and $y$, and the multiplicative identity for integers (i.e., $+1$), determine whether or not $x$ and $y$ are inverses under multiplication.

**RULE 56**

Apply $\oplus$Rule 19, Chapter 5 to Rule 15, Chapter 5 and apply the derived rule to $x$ and $y$. 

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EXERCISES 56
A. $-3$ and $+3$
B. $+6$ and $-2$
C. $+1$ and $-1$
D. $+1$ and $-4$

TASK 57
Given an integer $x$, show it is its own right inverse for subtraction.

RULE 57
Observe that $x - x = +0$ (Rule 26). (This shows that $x$ is its own right inverse.)

EXAMPLES 57
A. Given: $+7$
Answer:
The right inverse of $+7$ is $+7$ since $+7 - +7 = +0$.
B. Given: $-6$
Answer:
The right inverse of $-6$ is $-6$ since $-6 - -6 = +0$

EXERCISES 57
A. $+9$
B. $-7$
C. $+5$
D. $-8$
E. $+2$

√TASK 58
Given an integer $x$, show that it is its own right inverse for division.
Chapter 8
Section 8.4

RULE 58

Apply Rule 19, Chapter 5 to Rule 57 and apply the derived rule to x.

EXERCISES 58

A. $-3$
B. $+\frac{3}{4}$
C. $-11$
D. $+7$
E. $-3$
8.5. **Distributivity**

**TASK 59**

Given three integers \( x, y, \) and \( z \), determine whether or not the distributive property of multiplication over addition holds for \( x, y, \) and \( z \).

**RULE 59**

Apply Rule 19, Chapter 5 to Rule 36, Chapter 5 and apply the derived rule to \( x, y, \) and \( z \).

**EXERCISES 59**

A. \(-7, +6, +4\)
B. \(+3, -4, -6\)
C. \(-8, -2, +4\)
D. \(+6, +7, -3\)
E. \(+5, +2, +4\)

**TASK 60**

Given three integers \( x, y, \) and \( z \), determine whether or not the distributive property of multiplication over subtraction holds for \( x, y, \) and \( z \).

**RULE 60**

Apply Rule 19, Chapter 5 to Rule 36, Chapter 5 and apply the derived rule to \( x, y, \) and \( z \).

**EXERCISES 60**

A. \(-4, -3, +6\)
B. \(+5, +7, +2\)
C. \(+3, -4, -8\)
D. \(-2, -5, -9\)
E. \(+6, +5, +2\)

**TASK 61**

Given three integers \( x, y, \) and \( z \), determine whether or not the distributive property of division over addition holds for \( x, y, \) and \( z \).
RULE 61

Apply Rule 19, Chapter 5 to Rule 36, Chapter 5 and apply the derived rule to x, y, and z.

EXERCISES 61

A. -12, +6, -3
B. +10, -4, +1
C. +16, -8, -4
D. -28, -7, +4
E. -36, -9, +3
8.6. **Multiplication by +0**

**TASK 62**

Given an integer $x$, and the additive identity $+0$, show that $x \times +0 = +0$.

**RULE 62**

Apply Rule 29 to $x$ and $+0$.

**EXERCISES 62**

A. $-4$
B. $+7$
C. $-6$
D. $+3$
E. $-5$
Chapter 8

Other Properties: Relationships Between the Natural Numbers.

8.7. Positive Rationals, and Integers

**TASK 63**

Given (1) a rule which shows that a correspondence between two systems preserves operations on corresponding elements of the systems, (2) two new systems, and (3) a new correspondence; generate a rule for showing that the new correspondence between the new systems preserves the operations on corresponding elements of the new systems.

**RULE 63**

In the given rule, replace the original operations with the new operations, the original elements with elements of the new systems, and the original correspondence with the new correspondence.

**EXAMPLES 63**

A. Given: (1) Rule 72, Chapter 7, (2) the system of natural numbers under addition and the system of integers under addition, and (3) the correspondence \( n \leftrightarrow \overline{n} \).

Answer:

Find \( k + m \) using the appropriate rule (Rule 8, Chapter 5) for natural numbers. Next find \( \overline{k} + \overline{m} \) using the appropriate rule (Rule 15, Chapter 8) for integers. Observe that \( k + m \leftrightarrow \overline{k} + \overline{m} \).

B. Given: (1) Rule 72, Chapter 7, (2) the system of natural numbers under multiplication, and the system of integers under multiplication, and (3) the correspondence \( n \leftrightarrow \overline{n} \).

Answer:

Find \( k \times m \) using the appropriate rule (Rule 22, Chapter 5) for natural numbers. Next, find \( \overline{k} \times \overline{m} \) using the appropriate rule (Rule 29, Chapter 8) for integers. Observe that \( k \times m \leftrightarrow \overline{k} \times \overline{m} \).

**EXERCISES 63**

A. (1) Rule 72, Chapter 7, (2) the system of natural numbers under subtraction and the system of integers under subtraction, and (3) the correspondence \( n \leftrightarrow \overline{n} \).

B. (1) Rule 72, Chapter 7, (2) the system of natural numbers under subtraction and the system of integers under subtraction, and (3) the correspondence \( n \leftrightarrow \overline{n} \).
C. (1) Rule 72, Chapter 7, (2) the system of natural numbers under addition and the system of integers under addition and (3) the correspondence \( n \leftrightarrow +n \).

D. (1) Rule 72, Chapter 7, (2) the system of natural numbers under division and the system of integers under division, and (3) the correspondence \( n \leftrightarrow +n \).

**Task 64**

Given (1) two elements of a system, (2) a correspondence between that system and a second system, and (3) an operation within each system, show that the correspondence preserves the operation (for the given elements).

**Rule 64**

Apply Rule 63 to Rule 72, Chapter 7. Apply the derived rule to the given elements.

**Examples 64**

A. Given: (1) Natural numbers 8 and 9, (2) the correspondence \( n \leftrightarrow +n \) between natural numbers and integers, and (3) the operation of addition.

Answer:

Apply Rule 63 to Rule 72, Chapter 7 (Example 63A) and apply the derived rule to 8 and 9. \( 8 + 9 = 17, +8 + +9 = +17 \). Observe that \( 8 + 9 = 17 \leftrightarrow +8 + +9 = +17 \).

B. Given: Natural numbers 7 and 13, the correspondence \( n \leftrightarrow +n \) between natural numbers and integers and the operation of multiplication.

Answer:

Apply Rule 63 to Rule 72, Chapter 7 (Example 63B) and apply the derived rule to 7 and 13. \( 7 \times 13 = 91, +7 \times +13 = +91 \). Observe \( 7 \times 13 = 91 \leftrightarrow +7 \times +13 = +91 \).

**Exercises 64**

A. Natural numbers 7 and 3, the correspondence \( n \leftrightarrow +n \) (between natural numbers and integers), and the operation subtraction (see Exercise 63A).

B. Natural numbers 7 and 3, the correspondence \( n \leftrightarrow +n \), and the operation of subtraction (see Exercise 63B).
C. Natural numbers 9 and 19, the correspondence $n \leftrightarrow n$, and the operation of addition. (see Exercise 63C).

D. Natural numbers 8 and 4, the correspondence $n \leftrightarrow n$, and the operation of division (see Exercise 63D).

---

**TASK 65**

Given a 1-1 correspondence $n \leftrightarrow x$, between the natural numbers and the (positive and negative) integers and a 1-1 correspondence $n \leftrightarrow \mathbb{N}$, between the natural numbers and the positive rational numbers (Rule 73, Chapter 7), describe how this induces a 1-1 correspondence between the integers and the positive rationals.

**ANSWER 65**

A 1-1 correspondence between the integers and the positive rationals is induced by letting the integer $x$ correspond to that positive rational number $r$ which corresponds to the same natural number $n$ as does $x$. (i.e., $x \leftrightarrow r$).

---

**TASK 66**

Given the integers $x$ and $y$, an operation $\star$, and the correspondence $\mathbb{N} \leftrightarrow \mathbb{Q}$, between integers and positive rationals (as obtained through Answer 65 and Rule 73, Chapter 7), show that the correspondence does not preserve the operations.

**RULE 66**

Find the two positive rational number $\frac{r_x}{r_y}$ and $\frac{r_x}{r_y}$ corresponding to $x$ and $y$. Find $x \star y$ and $\frac{r_x}{r_y} \neq \frac{r_x}{r_y}$. Observe that $x \star y \neq \frac{r_x}{r_y} \neq \frac{r_x}{r_y}$. Therefore, the correspondence does not preserve the operations.

**EXAMPLES 66**

A. Given: $+2$, $-2$ and $\times$.

Answer:

$+2 \leftrightarrow \frac{3}{1}$ and $-2 \leftrightarrow \frac{2}{3}$. $+2 + -2 = +0$ and $+0 \leftrightarrow \frac{1}{1}$, but $\frac{3}{1} + \frac{2}{3} = \frac{11}{3}$
Therefore, \[+2 + 2 = \frac{3}{1} + \frac{2}{3}\]

B. Given: \(-1, +1\) and \(x\)

Answer:

\[-1 \leftrightarrow \frac{2}{1} \quad \text{and} \quad +1 \leftrightarrow \frac{1}{2}. \quad -1 \times +1 = -1 \quad \text{and} \quad -1 \leftrightarrow \frac{2}{1} \quad \text{but}\]

\[
\frac{2}{1} \times \frac{1}{2} = \frac{1}{1}
\]
Therefore, \(-1 \times +1 = \frac{2}{1} \times \frac{1}{2}\)

EXERCISES 66

A. \(-1, -1, \text{and } +\)
B. \(+1, -2, \text{and } x\)
C. \(-3, +1, \text{and } +\)
D. \(-1, -2, \text{and } x\)
E. \(+0, +2, \text{and } +\)

TASK 67

Justify the statement "the set of integers is not dense."

ANSWER 67

The set of integers is not dense because there is no integer between two consecutive integers (e.g., \(+7\) and \(+8\)).

TASK 68

Given an integer \(x\), find its unique successor.

RULE 68

Add \(+1\) to \(x\) to find the unique successor of \(x\).

EXAMPLES 68

A. Given: \(-7\).
Answer:

The unique successor of \(-7\) is \(-7 + 1 = -6\).

B. Given: \(+4\).

Answer:

The unique successor of \(+4\) is \(+4 + 1 = +5\).

EXERCISES 68

A. \(-13\)
B. \(+9\)
C. \(-8\)
D. \(+14\)
E. \(-11\)

---

TASK 69

Given an integer \(x\), find its unique predecessor.

RULE 69

Add \(-1\) to \(x\) to find the unique predecessor of \(x\).

EXAMPLES 69

A. Given: \(-5\)

Answer:

The unique predecessor of \(-5\) is \(-5 - 1 = -6\).

B. Given: \(+8\)

Answer:

The unique predecessor of \(+8\) is \(+8 - 1 = +7\).
EXERCISES 69

A. +4
B. -15
C. +9
D. -1
E. +1

TASK 70

Justify the statement "the set of integers does not have a least element."

ANSWER 70

Given any integer we can always find one smaller because each integer has a predecessor (Rule 69). Hence there can be no least integer.
CHAPTER 9

FURTHER EXTENSIONS

SECTION 1. The System of Signed Rationals

TASK 1

Given two signed fractions x/y and x'/y', determine whether or not they are equivalent.

RULE 1

Apply Rule 17, Chapter 7. (Note: This rule also applies to integer pairs.)

EXAMPLES 1

A. Given: +3/-4, -27/+36.

Answer:

+3/-4 is equivalent to -27/+36 because +3 • +36 = -4 • -27 (= +108).

B. Given: -2/-5, -4/+10

Answer:

-2/-5 is not equivalent to -4/+10 because -2 • +10 = -20 and -5 • -4 = +20. Therefore, -2 • +10 ≠ -5 • -4.

EXERCISES 1

A. -4/+7, +12/-21
B. +5/-6, -3/+4
C. +9/+11, -27/-33
D. -2/-3, +8/+12
E. $^+1/^-4$, $^+3/^-12$

---

**TASK 2**

Given a signed fraction $x/y$, write the signed rational represented by $x/y$.

**RULE 2**

Reduce $x/y$ to lowest terms (Rule 19, Chapter 7). (Suppose $b/c$ is the unsigned magnitude of the fraction in lowest terms). If

(a) $x$ and $y$ have like signs, form the set \[
\{ \frac{b}{c}, \frac{-b}{c}, \frac{+2b}{2c}, \frac{-2b}{2c}, \frac{+3b}{3c}, \frac{-3b}{3c}, \ldots \} \]

and denote this signed rational by $^+b/c$,

(b) $x$ and $y$ have unlike signs, form the set \[
\{ \frac{b}{c}, \frac{-b}{c}, \frac{+2b}{2c}, \frac{-2b}{2c}, \frac{+3b}{3c}, \frac{-3b}{3c}, \ldots \} \]

and denote this signed rational by $^-b/c$.

**EXAMPLES 2**

A. Given: $^+8/^-12$.

Answer:

\[
\frac{8}{12} = \frac{8}{2} \times \frac{1}{3} = \frac{4}{3} \left\{ \frac{+2}{3}, \frac{-2}{3}, \frac{+4}{3}, \frac{-4}{3}, \frac{+6}{3}, \frac{-6}{3}, \ldots \right\} = \frac{-2}{3}.
\]


Answer:

\[
\frac{4}{14} = \frac{2}{7} \times \frac{2}{7} = \frac{2}{7} \left\{ \frac{+2}{7}, \frac{-2}{7}, \frac{+4}{14}, \frac{-4}{14}, \frac{+6}{21}, \frac{-6}{21}, \ldots \right\} = \frac{+2}{7}.
\]

**EXERCISES 2**

A. $^-5/^+12$

B. $^+6/^-22$

C. $^-4/^+16$
TASK 3

Given two (signed) rational numbers $R_1$ and $R_2$, find the sum $R_1 + R_2$.

RULE 3

Apply Rule 27, Chapter 7 and Rules 18 and 33, Chapter 8 to find the fraction corresponding to the sum. Next apply Rule 2 to find the (signed) rational corresponding to this fraction.

EXAMPLES 3

A. Given: \( \left\{ \frac{-1}{2}, \frac{1}{4}, \frac{2}{4}, \frac{2}{4}, \ldots \right\} \) and \( \left\{ \frac{1}{4}, \frac{-1}{4}, \frac{2}{8}, \frac{-2}{8}, \ldots \right\} \).

Answer:


\[
\left( \frac{-1}{2} \right) \left( \frac{1}{4} \right) + \left( \frac{2}{4} \right) \left( \frac{2}{8} \right) = \frac{-16}{32} + \frac{8}{32} = \frac{-8}{32}.
\]

By Rule 2, \( \left\{ \frac{-1}{4}, \frac{1}{4}, \frac{2}{8}, \frac{-2}{8}, \ldots \right\} = -1/4 \).

B. Given: \( \left\{ \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{-1}{5}, \ldots \right\} \) and \( \left\{ \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \ldots \right\} \).

Answer:

Choose representatives $+6/10$ and $-4/-6$.

\[
\left( \frac{1}{5} \right) \left( \frac{-1}{5} \right) + \left( \frac{1}{10} \right) \left( \frac{1}{10} \right) = \frac{-36}{60} + \frac{60}{60} = \frac{76}{60}.
\]

By Rule 2, \( \left\{ \frac{-19}{15}, \frac{-19}{15}, \frac{19}{15}, \frac{-19}{15}, \ldots \right\} = \frac{19}{15} \).

EXERCISES 3

A. \( \left\{ \frac{-1}{3}, \frac{1}{3}, \frac{-2}{6}, \frac{2}{6}, \ldots \right\} \) and \( \left\{ \frac{1}{4}, \frac{-1}{4}, \frac{2}{8}, \frac{-2}{8}, \ldots \right\} \)

B. \( \left\{ \frac{1}{6}, \frac{-1}{6}, \frac{10}{12}, \frac{-10}{12}, \ldots \right\} \) and \( \left\{ \frac{1}{4}, \frac{-1}{4}, \frac{2}{8}, \frac{-2}{8}, \ldots \right\} \)

C. \( \left\{ \frac{-7}{3}, \frac{-7}{3}, \frac{-14}{6}, \frac{-14}{6}, \ldots \right\} \) and \( \left\{ \frac{-2}{3}, \frac{2}{3}, \frac{-4}{6}, \frac{4}{6}, \ldots \right\} \)

D. \( \left\{ \frac{-1}{6}, \frac{1}{6}, \frac{2}{12}, \frac{-2}{12}, \ldots \right\} \) and \( \left\{ \frac{5}{2}, \frac{5}{2}, \frac{-10}{4}, \frac{-10}{4}, \ldots \right\} \)
TASK 4

Given two (signed) rationals \( R_1 \) and \( R_2 \), find the product \( R_1 \times R_2 \).

RULE 4

Apply Rule 42, Chapter 7 and Rule 33, Chapter 8 to find the fraction corresponding to the product. Next apply Rule 2 to find the (signed) rational corresponding to this fraction.

EXAMPLES 4

A. Given: \[ \left\{ \frac{-1}{3}, \frac{-1}{3}, \frac{-2}{6}, \frac{-2}{6}, \ldots \right\} \text{ and } \left\{ \frac{+2}{5}, \frac{-2}{5}, \frac{+4}{10}, \frac{-4}{10}, \ldots \right\} \].

Answer:

Choose representatives \(-1/3\) and \(+4/10\).

\[ \frac{(-1)(+4)}{(-3)(+10)} = \frac{-4}{-30} \]

By Rule 2, \( \left\{ \frac{-2}{15}, \frac{-2}{15}, \frac{-2}{30}, \frac{-2}{30}, \ldots \right\} = -2/15 \).

B. Given: \( \left\{ \frac{+3}{7}, \frac{-3}{7}, \frac{+6}{14}, \frac{-6}{14}, \ldots \right\} \text{ and } \left\{ \frac{+5}{3}, \frac{-5}{3}, \frac{+10}{6}, \frac{-10}{6}, \ldots \right\} \).

Answer:

Choose representatives \(-3/7\) and \(+10/6\).

\[ \frac{(-3)(+10)}{(-7)(+6)} = \frac{-30}{-42} \]

By Rule 2 \( \left\{ \frac{+5}{7}, \frac{-5}{7}, \frac{+10}{14}, \frac{-10}{14}, \ldots \right\} = +5/7 \).

EXERCISES 4

A. \( \left\{ \frac{+4}{9}, \frac{-4}{9}, \frac{+8}{18}, \frac{-8}{18}, \ldots \right\} \), \( \left\{ \frac{-1}{2}, \frac{-1}{2}, \frac{+2}{4}, \frac{-2}{4}, \ldots \right\} \)

B. \( \left\{ \frac{-6}{5}, \frac{-6}{5}, \frac{+12}{10}, \frac{-12}{10}, \ldots \right\} \), \( \left\{ \frac{+7}{3}, \frac{-7}{3}, \frac{+14}{6}, \frac{-14}{6}, \ldots \right\} \)

C. \( \left\{ \frac{-12}{7}, \frac{+12}{7}, \frac{-24}{14}, \frac{+24}{14}, \ldots \right\} \), \( \left\{ \frac{+8}{6}, \frac{-8}{6}, \frac{+10}{12}, \frac{-10}{12}, \ldots \right\} \)

D. \( \left\{ \frac{+3}{4}, \frac{-3}{4}, \frac{+6}{8}, \frac{-6}{8}, \ldots \right\} \), \( \left\{ \frac{+5}{2}, \frac{-5}{2}, \frac{+10}{4}, \frac{-10}{4}, \ldots \right\} \)
**TASK 5**

Given two (signed) rationals $R_1$ and $R_2$, find the difference $R_1 - R_2$.

**RULE 5**

Apply Rule 34, Chapter 7 and Rules 26 and 33, Chapter 8 to find the fraction corresponding to the difference. Next apply Rule 2 to find the (signed) rational corresponding to this fraction.

**EXAMPLES 5**

A. Given: \( \left\{ \frac{1}{6}, \frac{-1}{6}, \frac{-2}{12}, \frac{-2}{12}, \ldots \right\} \) and \( \left\{ \frac{-7}{8}, \frac{-7}{8}, \frac{+14}{16}, \frac{-14}{16}, \ldots \right\} \).

Answer:

Choose representatives $-1/6$ and $+14/16$.

\[
\frac{-1}{6} \left( \frac{1+16}{1+16} \right) - \frac{1}{6} \left( \frac{+6}{+16} \right) = \frac{-16 + 84}{-96} = \frac{-100}{96}
\]

By Rule 2, \( \left\{ \frac{+25}{24}, \frac{-25}{24}, \frac{+50}{48}, \frac{-50}{48}, \ldots \right\} = \frac{-25}{24} \).

B. Given: \( \left\{ \frac{-5}{13}, \frac{-5}{13}, \frac{+10}{16}, \frac{-10}{16}, \ldots \right\} \) and \( \left\{ \frac{-7}{12}, \frac{-7}{12}, \frac{+14}{12}, \frac{-14}{12}, \ldots \right\} \).

Answer:

Choose representatives $-5/3$ and $+7/12$.

\[
\frac{-1}{3} \left( \frac{+2}{2} \right) - \frac{-1}{3} \left( \frac{+7}{12} \right) = \frac{-10 - 21}{-6} = \frac{-11}{-6}
\]

By Rule 2, \( \left\{ \frac{+11}{-6}, \frac{-11}{-6}, \frac{+22}{12}, \frac{-22}{12}, \ldots \right\} = \frac{11}{6} \).

**EXERCISES 5**

A. \( \left\{ \frac{+4}{10}, \frac{-4}{10}, \frac{+8}{10}, \frac{-8}{10}, \ldots \right\}, \left\{ \frac{+4}{9}, \frac{-4}{9}, \frac{+8}{9}, \frac{-8}{9}, \ldots \right\} \)

B. \( \left\{ \frac{+6}{7}, \frac{-6}{7}, \frac{+12}{14}, \frac{-12}{14}, \ldots \right\}, \left\{ \frac{+3}{7}, \frac{-3}{7}, \frac{+6}{10}, \frac{-6}{10}, \ldots \right\} \)

C. \( \left\{ \frac{+9}{2}, \frac{-9}{2}, \frac{+18}{4}, \frac{-18}{4}, \ldots \right\}, \left\{ \frac{+2}{3}, \frac{-2}{3}, \frac{+4}{6}, \frac{-4}{6}, \ldots \right\} \)

D. \( \left\{ \frac{-3}{10}, \frac{-3}{10}, \frac{+6}{20}, \frac{-6}{20}, \ldots \right\}, \left\{ \frac{+1}{3}, \frac{-1}{3}, \frac{+2}{6}, \frac{-2}{6}, \ldots \right\} \)
TASK 6

Given two (signed) rationals $R_1$ and $R_2 (R_2 \neq 0)$, find the quotient $R_1 + R_2$.

RULE 6

Apply Rule 51, Chapter 7 and Rule 33, Chapter 8 to find the fraction corresponding to the quotient. Next apply Rule 2 to find the (signed) rational corresponding to this fraction.

EXAMPLES 6

A. Given: $\left\{ \frac{+1}{3}, \frac{-1}{3}, \frac{+2}{6}, \frac{-2}{6}, \ldots \right\}$ and $\left\{ \frac{+5}{3}, \frac{-5}{3}, \frac{+10}{6}, \frac{-10}{6}, \ldots \right\}$.

Answer:

Choose representatives $\frac{-1}{3}$ and $\frac{+10}{6}$.

\[
\frac{(-1)(+6)}{(+3)(+10)} = \frac{-6}{+30}
\]

By Rule 2, $\left\{ \frac{+1}{5}, \frac{-1}{5}, \frac{+2}{10}, \frac{-2}{10}, \ldots \right\} = -1/5$.

B. Given: $\left\{ \frac{+3}{2}, \frac{-3}{2}, \frac{+6}{4}, \frac{-6}{4}, \ldots \right\}$ and $\left\{ \frac{+6}{5}, \frac{-6}{5}, \frac{+12}{10}, \frac{-12}{10}, \ldots \right\}$.

Answer:

Choose representatives $\frac{-3}{2}$ and $\frac{-12}{10}$.

\[
\frac{(-3)(-10)}{(-2)(-12)} = \frac{+30}{-24}
\]

By Rule 2, $\left\{ \frac{+6}{4}, \frac{-5}{4}, \frac{+10}{8}, \frac{-10}{8}, \ldots \right\} = +5/4$.

EXERCISES 6

A. $\left\{ \frac{+1}{3}, \frac{-1}{3}, \frac{+2}{6}, \frac{-2}{6}, \ldots \right\}$, $\left\{ \frac{+1}{4}, \frac{-1}{4}, \frac{+2}{8}, \frac{-2}{8}, \ldots \right\}$

B. $\left\{ \frac{+8}{9}, \frac{-8}{9}, \frac{+10}{16}, \frac{-10}{16}, \ldots \right\}$, $\left\{ \frac{+11}{4}, \frac{-11}{4}, \frac{+22}{8}, \frac{-22}{8}, \ldots \right\}$

C. $\left\{ \frac{+8}{9}, \frac{-8}{9}, \frac{+16}{18}, \frac{-16}{18}, \ldots \right\}$, $\left\{ \frac{+8}{9}, \frac{-8}{9}, \frac{+16}{18}, \frac{-16}{18}, \ldots \right\}$

D. $\left\{ \frac{+3}{5}, \frac{-3}{5}, \frac{+6}{10}, \frac{-6}{10}, \ldots \right\}$, $\left\{ \frac{+1}{2}, \frac{-1}{2}, \frac{+2}{4}, \frac{-2}{4}, \ldots \right\}$
\[ \text{Task 7} \]

Given two (signed) rational numbers \( R_1 \) and \( R_2 \), show that the operation of addition of rational numbers is well-defined for \( R_1 \) and \( R_2 \).

\[ \text{Rule 7} \]

Apply Rule 23, Chapter 5 to \( R_1 \) and \( R_2 \).

\[ \text{Exercises 7} \]

A. \( \{ \frac{3}{4}, \frac{3}{4}, \frac{6}{8}, \frac{6}{8}, \ldots \}, \{ \frac{4}{7}, \frac{4}{7}, \frac{8}{14}, \frac{8}{14}, \ldots \} \)

B. \( \{ \frac{9}{2}, \frac{9}{2}, \frac{18}{4}, \frac{18}{4}, \ldots \}, \{ \frac{8}{3}, \frac{8}{3}, \frac{16}{6}, \frac{16}{6}, \ldots \} \)

C. \( \{ \frac{3}{7}, \frac{3}{7}, \frac{30}{14}, \frac{30}{14}, \ldots \}, \{ \frac{5}{9}, \frac{5}{9}, \frac{10}{18}, \frac{10}{18}, \ldots \} \)

D. \( \{ \frac{8}{5}, \frac{8}{5}, \frac{16}{10}, \frac{16}{10}, \ldots \}, \{ \frac{9}{5}, \frac{9}{5}, \frac{18}{10}, \frac{18}{10}, \ldots \} \)

\[ \text{Task 8} \]

Given two (signed) rational numbers \( R_1 \) and \( R_2 \), show that the operation of subtraction of rational numbers is well-defined for \( R_1 \) and \( R_2 \).

\[ \text{Rule 8} \]

Apply Rule 23, Chapter 5 to \( R_1 \) and \( R_2 \).

\[ \text{Exercises 8} \]

A. \( \{ \frac{3}{4}, \frac{3}{4}, \frac{6}{8}, \frac{6}{8}, \ldots \}, \{ \frac{5}{4}, \frac{5}{4}, \frac{10}{8}, \frac{10}{8}, \ldots \} \)

B. \( \{ \frac{1}{2}, \frac{1}{2}, \frac{2}{4}, \frac{2}{4}, \ldots \}, \{ \frac{1}{6}, \frac{1}{6}, \frac{2}{12}, \frac{2}{12}, \ldots \} \)

C. \( \{ \frac{11}{4}, \frac{11}{4}, \frac{22}{8}, \frac{22}{8}, \ldots \}, \{ \frac{5}{8}, \frac{5}{8}, \frac{10}{16}, \frac{10}{16}, \ldots \} \)

D. \( \{ \frac{7}{6}, \frac{7}{6}, \frac{14}{12}, \frac{14}{12}, \ldots \}, \{ \frac{2}{3}, \frac{2}{3}, \frac{4}{6}, \frac{4}{6}, \ldots \} \)
\( \sqrt{\text{TASK 9}} \)

Given two (signed) rational numbers \( R_1 \) and \( R_2 \), show that the operation of multiplication of rational numbers is well-defined for \( R_1 \) and \( R_2 \).

**RULE 9**

Apply \( \text{Rule 23, Chapter 5} \) to \( R_1 \) and \( R_2 \).

**EXERCISES 9**

A. \( \{ \frac{9}{7}, \frac{-9}{7}, \frac{18}{14}, \frac{-18}{14}, \ldots \} \), \( \{ \frac{2}{3}, \frac{-2}{3}, \frac{4}{6}, \frac{-4}{6}, \ldots \} \)

B. \( \{ \frac{-18}{5}, \frac{18}{5}, \frac{-36}{10}, \frac{36}{10}, \ldots \} \), \( \{ \frac{5}{6}, \frac{-5}{6}, \frac{10}{12}, \frac{-10}{12}, \ldots \} \)

C. \( \{ \frac{-8}{9}, \frac{8}{9}, \frac{16}{18}, \frac{-16}{18}, \ldots \} \), \( \{ \frac{4}{3}, \frac{-4}{3}, \frac{6}{9}, \frac{-6}{9}, \ldots \} \)

D. \( \{ \frac{5}{9}, \frac{-5}{9}, \frac{10}{16}, \frac{-10}{16}, \ldots \} \), \( \{ \frac{5}{10}, \frac{-5}{10}, \frac{-24}{12}, \frac{24}{12}, \ldots \} \)

\( \sqrt{\text{TASK 10}} \)

Given two (signed) rational numbers \( R_1 \) and \( R_2 \neq 0 \), show the operation of division of rational numbers is well-defined for \( R_1 \) and \( R_2 \).

**RULE 10**

Apply \( \text{Rule 23, Chapter 5} \) to \( R_1 \) and \( R_2 \).

**EXERCISES 10**

A. \( \{ \frac{3}{4}, \frac{-3}{4}, \frac{6}{8}, \frac{-6}{8}, \ldots \} \), \( \{ \frac{3}{8}, \frac{-3}{8}, \frac{6}{16}, \frac{-6}{16}, \ldots \} \)

B. \( \{ \frac{1}{3}, \frac{-1}{3}, \frac{2}{5}, \frac{-2}{5}, \ldots \} \), \( \{ \frac{5}{6}, \frac{-5}{6}, \frac{10}{12}, \frac{-10}{12}, \ldots \} \)

C. \( \{ \frac{3}{2}, \frac{-3}{2}, \frac{6}{4}, \frac{-6}{4}, \ldots \} \), \( \{ \frac{1}{8}, \frac{-1}{8}, \frac{2}{16}, \frac{-2}{16}, \ldots \} \)

D. \( \{ \frac{3}{7}, \frac{-3}{7}, \frac{6}{14}, \frac{-6}{14}, \ldots \} \), \( \{ \frac{2}{5}, \frac{-2}{5}, \frac{4}{10}, \frac{-4}{10}, \ldots \} \)

\( \sqrt{\text{TASK 11}} \)

Given a (signed) rational \( x/y \), represent it as an equivalence class of ordered pairs of non-negative rationals.
RULE 11

If the sign of $x/y$ is $+$, write the equivalence class as $\{(a/b, c/d) \mid a/b - c/d = x/y\}$. If the sign of $x/y$ is $-$, write the equivalence class as $\{(a/b, c/d) \mid c/d - a/b = x/y\}$. (In each case $a/b$ and $c/d$ are non-negative rationals.)

EXAMPLES 11


Answer:

$\{(a/b, c/d) \mid c/d - a/b = 3/4\}$.

B. Given: $+7/6$.

Answer:

$\{(a/b, c/d) \mid a/b - c/d = 7/6\}$.

EXERCISES 11

A. $-2/5$
B. $+9/2$
C. $-4/7$
D. $+6/11$

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TASK 12

Given an ordered pair of non-negative rationals $(a/b, c/d)$ (i.e., given an element from an equivalence class of ordered pairs of non-negative rationals), find the rational it represents.

RULE 12

Find the difference $e'/f'$ of $a/b$ and $c/d$ (larger-smaller) and reduce the difference to lowest terms, $e/f$. If $a/b$ is less than $c/d$, write $-e/f$; otherwise write $+e/f$.

EXAMPLES 12

A. Given: $(3/5, 2/3)$.

Answer:

$2/3 - 3/5 = 1/15$; therefore $(3/5, 2/3)$ represents $-1/15$. 341
B. Given: \((3/2, 2/7)\).

Answer:

\[
\frac{3}{2} - \frac{2}{7} = \frac{17}{14}; \text{ therefore } (3/2, 2/7) \text{ represents } +\frac{17}{14}.
\]

EXERCISES 12

A. \((4/3, 2/5)\)

B. \((1/6, 3/7)\)

C. \((4/5, 3/8)\)

D. \((3/10, 1/3)\)

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TASK 13

Given two rationals \(R_1\) and \(R_2\) (expressed as equivalence classes of ordered pairs of non-negative rationals), find the sum \(R_1 + R_2\).

RULE 13

Choose representatives \((a/b, c/d)\) of \(R_1\) and \((a'/b', c'/d')\) of \(R_2\). Find \((a/b + a'/b', c/d + c'/d')\). Apply Rule 12 to find the rational corresponding to this ordered pair.

EXAMPLES 13

A. Given: \(R_1 = \{(a/b, c/d) \mid a/b - c/d = 3/4\}\) and \(R_2 = \{(a/b, c/d) \mid c/d - a/b = 2/3\}\).

Answer:

Choose representatives \((5/4, 2/4)\) and \((2/3, 4/3)\) of \(R_1\) and \(R_2\) respectively. 
\((5/4 + 2/3, 2/4 + 4/3) = (23/12, 22/12)\). By Rule 12, \(R_1 + R_2 = +\frac{1}{12}\).

B. Given: \(R_1 = \{(a/b, c/d) \mid c/d - a/b = 1/6\}\) and \(R_2 = \{(a/b, c/d) \mid c/d - a/b = 3/2\}\).

Answer:

Choose representatives \((6/6, 7/6)\) and \((2/2, 5/2)\) of \(R_1\) and \(R_2\) respectively. 
\((6/6 + 2/2, 7/6 + 5/2) = (24/12, 44/12)\). By Rule 12, \(R_1 + R_2 = -\frac{5}{3}\).
EXERCISES 13

A. \{(a/b, c/d) \mid a/b - c/d = 2/3\}, \{(a/b, c/d) \mid a/b - c/d = 3/5\}
B. \{(a/b, c/d) \mid c/d - a/b = 1/5\}, \{(a/b, c/d) \mid a/b - c/d = 5/2\}
C. \{(a/b, c/d) \mid c/d - a/b = 1/3\}, \{(a/b, c/d) \mid c/d - a/b = 5/6\}
D. \{(a/b, c/d) \mid a/b - c/d = 4/3\}, \{(a/b, c/d) \mid c/d - a/b = 1/6\}

TASK 14

Given two rationals \(R_1\) and \(R_2\) (expressed as equivalence classes of ordered pairs of non-negative rationals), find the product \(R_1 \times R_2\).

RULE 14

Choose representatives \((a/b, c/d)\) of \(R_1\) and \((a'/b', c'/d')\) of \(R_2\). Find \((a/b \cdot a'/b') + (c/d \cdot c'/d'), (a/b \cdot c'/d') + (c/d \cdot a'/b')\). Apply Rule 12 to find the rational corresponding to this ordered pair.

EXAMPLES 14

A. Given: \(R_1 = \{(a/b, c/d) \mid a/b - c/d = 1/5\}\) and \(R_2 = \{(a/b, c/d) \mid a/b - c/d = 5/6\}\).

Answer:

Choose representatives \((3/5, 2/5)\) and \((7/6, 2/6)\) of \(R_1\) and \(R_2\) respectively.

\((3/5 \cdot 2/5 + 7/6 \cdot 2/6, 3/5 \cdot 2/6 + 7/6 \cdot 2/5) = (7/10 + 2/15, 1/2 + 7/15)\)
\((25/30, 20/30) = (7/10, 13/15)\).

B. Given: \(R'_1 = \{(a/b, c/d) \mid c/d - a/b = 2/3\}\) and \(R'_2 = \{(a/b, c/d) \mid a/b - c/d = 3/4\}\).

Answer:

Choose representatives \((1/3, 3/3)\) and \((4/4, 1/4)\) of \(R'_1\) and \(R'_2\) respectively.

\((1/3 \cdot 4/4 + 3/3 \cdot 1/4, 1/3 \cdot 1/4 + 3/3 \cdot 4/4) = (4/12 + 3/12, 1/12 + 12/12)\)
\((7/12, 13/12)\).

By Rule 12, \(R'_1 \times R'_2 = 1/2\).
EXERCISES 14

A. \{(a/b, c/d) \mid a/b - c/d = 1/4\}, \{(a/b, c/d) \mid a/b - c/d = 1/3\}
B. \{(a/b, c/d) \mid c/d - a/b = 2/5\}, \{(a/b, c/d) \mid a/b - c/d = 5/3\}
C. \{(a/b, c/d) \mid c/d - a/b = 3/2\}, \{(a/b, c/d) \mid c/d - a/b = 5/6\}
D. \{(a/b, c/d) \mid a/b - c/d = 2/5\}, \{(a/b, c/d) \mid c/d - a/b = 3/4\}

TASK 15

Given two rationals \(R_1\) and \(R_2\) (as in Tasks 13, 14) find the difference \(R_1 - R_2\).

RULE 15

Choose representatives \((a/b, c/d)\) and \((a'/b', c'/d')\) of \(R_1\) and \(R_2\) respectively such that \(a/b > a'/b'\) and \(c/d > c'/d'\). Find \((a/b - a'/b', c/d - c'/d')\). Apply Rule 12 to find the rational corresponding to this ordered pair.

EXAMPLES 15

A. Given: \(R_1 = \{(a/b, c/d) \mid a/b - c/d = 2/3\}\) and \(R_2 = \{(a/b, c/d) \mid a/b - c/d = 1/6\}\).

Answer:

Choose representatives \((3/3, 1/3)\) and \((2/6, 1/6)\) of \(R_1\) and \(R_2\) respectively. 
\((3/3 - 2/6, 1/3 - 1/6) = (4/6, 1/6)\). By Rule 12, \(R_1 - R_2 = 1/2\).

B. Given: \(R_1 = \{(a/b, c/d) \mid c/d - a/b = 2/5\}\) and \(R_2 = \{(a/b, c/d) \mid a/b - c/d = 1/3\}\).

Answer:

Choose representatives \((5/5, 7/5)\) and \((2/3, 1/3)\) of \(R_1\) and \(R_2\) respectively. 
\((5/5 - 2/3, 7/5 - 1/3) = (5/15, 16/15)\). By Rule 12, \(R_1 - R_2 = 11/15\).

EXERCISES 15

A. \{(a/b, c/d) \mid a/b - c/d = 1/5\}, \{(a/b, c/d) \mid a/b - c/d = 3/4\}
B. \{(a/b, c/d) \mid a/b - c/d = 3/2\}, \{(a/b, c/d) \mid c/d - a/b = 3/5\}
C. \{(a/b, c/d) \mid c/d - a/b = 1/4\}, \{(a/b, c/d) \mid a/b - c/d = 2/7\}
D. \{(a/b, c/d) \mid c/d - a/b = 5/6\}, \{(a/b, c/d) \mid c/d - a/b = 2/3\}
**Chapter 9**

**Section 1**

**Task 16**

Given two rationals \( R_1 \) and \( R_2 \) (as in Tasks 13 - 15), find the quotient \( R_1 + R_2 \) (\( R_2 \neq 0 \)).

**Rule 16**

Choose representatives \((a/b, c/d)\) and \((a'/b', c'/d')\) of \( R_1 \) and \( R_2 \) respectively. Find \( Z = |a/b - c/d| + |a'/b' - c'/d'| \). Write \((P \cdot Z, (1 - P) \cdot Z)\) where \( P = 1 \) if both \( a/b \) and \( a'/b' \) are greater (less) than \( c/d \) and \( c'/d' \) respectively. Write \( P = 0 \) if \( a/b \) is greater (less) than \( c/d \) and \( a'/b' \) is less (greater) than \( c'/d' \). Apply Rule 12 to find the rational corresponding to this ordered pair.

**Examples 16**

A. Given: \( R_1 = \{(a/b, c/d) \mid a/b - c/d = 2/3\} \) and \( R_2 = \{(a/b, c/d) \mid c/d - a/b = 4/9\} \).

Answer:

Choose representatives \((4/3, 2/3)\) and \((1/9, 5/9)\) of \( R_1 \) and \( R_2 \) respectively.
\[ Z = |4/3 - 2/3| + |1/9 - 5/9| = 2/3 + 4/9 = 2/3 \cdot 9/4 = 6/4 = 3/2. \] Since \( 4/3 > 2/3 \) and \( 1/9 < 5/9 \), \( P = 0 \). Therefore the quotient is represented by \((0, 3/2)\). By Rule 12, \( R_1 + R_2 = -3/2 \).

B. Given: \( R_1 = \{(a/b, c/d) \mid c/d - a/b = 5/6\} \) and \( R_2 = \{(a/b, c/d) \mid c/d - a/b = 2/3\} \).

Answer:

Choose representatives \((1/6, 6/6)\) and \((2/3, 4/3)\) of \( R_1 \) and \( R_2 \) respectively.
\[ Z = |1/6 - 6/6| + |2/3 - 4/3| = 5/6 + 2/3 = 5/6 \cdot 3/2 = 5/4. \] Since \( 1/6 < 6/6 \) \( 2/3 < 4/3 \), \( P = 1 \). Therefore the quotient is represented by \((5/4, 0)\). By Rule 12, \( R_1 + R_2 = +5/4 \).

**Exercises 16**

A. \( \{(a/b, c/d) \mid a/b - c/d = 2/5\} \), \( \{(a/b, c/d) \mid a/b - c/d = 2/3\} \)

B. \( \{(a/b, c/d) \mid a/b - c/d = 4/3\} \), \( \{(a/b, c/d) \mid c/d - a/b = 5/6\} \)

C. \( \{(a/b, c/d) \mid c/d - a/b = 3/7\} \), \( \{(a/b, c/d) \mid c/d - a/b = 3/14\} \)

D. \( \{(a/b, c/d) \mid c/d - a/b = 2/9\} \), \( \{(a/b, c/d) \mid a/b - c/d = 5/3\} \)

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\textbf{VTASK 17}

Given two rational numbers \( R_1 \) and \( R_2 \) (as in Tasks 13-16), show that the operation of addition of rational numbers (as in Task 13) is well-defined for \( R_1 \) and \( R_2 \).

\textbf{RULE 17}

Apply Rule 23, Chapter 5 to \( R_1 \) and \( R_2 \).

\textbf{EXERCISES 17}

A. \( \{(a/b, c/d) \mid a/b - c/d = 2/5\}, \{(a/b, c/d) \mid a/b - c/d = 1/2\} \)
B. \( \{(a/b, c/d) \mid c/d - a/b = 3/7\}, \{(a/b, c/d) \mid a/b - c/d = 5/6\} \)
C. \( \{(a/b, c/d) \mid c/d - a/b = 3/2\}, \{(a/b, c/d) \mid c/d - a/b = 5/9\} \)
D. \( \{(a/b, c/d) \mid a/b - c/d = 1/4\}, \{(a/b, c/d) \mid c/d - a/b = 12/7\} \)

\textbf{VTASK 18}

Given two rational numbers \( R_1 \) and \( R_2 \) (as in Tasks 13-17), show that the operation of multiplication of rational numbers (as in Task 14) is well-defined for \( R_1 \) and \( R_2 \).

\textbf{RULE 18}

Apply Rule 23, Chapter 5 to \( R_1 \) and \( R_2 \).

\textbf{EXERCISES 18}

A. \( \{(a/b, c/d) \mid a/b - c/d = 1/5\}, \{(a/b, c/d) \mid a/b - c/d = 5/6\} \)
B. \( \{(a/b, c/d) \mid c/d - a/b = 2/3\}, \{(a/b, c/d) \mid a/b - c/d = 3/5\} \)
C. \( \{(a/b, c/d) \mid c/d - a/b = 1/3\}, \{(a/b, c/d) \mid c/d - a/b = 5/2\} \)
D. \( \{(a/b, c/d) \mid a/b - c/d = 5/3\}, \{(a/b, c/d) \mid c/d - a/b = 1/6\} \)

\textbf{VTASK 19}

Given two rational numbers \( R_1 \) and \( R_2 \) (as in Tasks 13-18), show that the operation of subtraction of rational numbers (as in Task 15) is well-defined for \( R_1 \) and \( R_2 \).

\textbf{RULE 19}

Apply Rule 23, Chapter 5 to \( R_1 \) and \( R_2 \).
EXERCISES 19

A. \{(a/b, c/d) | a/b - c/d = 5/3\}, \{(a/b, c/d) | a/b - c/d = 1/6\}
B. \{(a/b, c/d) | a/b - c/d = 2/5\}, \{(a/b, c/d) | c/d - a/b = 3/2\}
C. \{(a/b, c/d) | c/d - a/b = 5/9\}, \{(a/b, c/d) | c/d - a/b = 5/6\}
D. \{(a/b, c/d) | c/d - a/b = 3/4\}, \{(a/b, c/d) | a/b - c/d = 3/5\}

√TASK 20

Given two rational numbers R₁ and R₂ \(\neq 0\) (as in Tasks 13-19), show that the operation of division of rational numbers (as in Task 16) is well-defined for R₁ and R₂.

RULE 20

Apply Rule 23, Chapter 5 to R₁ and R₂.

EXERCISES 20

A. \{(a/b, c/d) | a/b - c/d = 1/3\}, \{(a/b, c/d) | a/b - c/d = 3/5\}
B. \{(a/b, c/d) | a/b - c/d = 6/5\}, \{(a/b, c/d) | c/d - a/b = 3/10\}
C. \{(a/b, c/d) | c/d - a/b = 7/3\}, \{(a/b, c/d) | c/d - a/b = 4/5\}
D. \{(a/b, c/d) | c/d - a/b = 1/2\}, \{(a/b, c/d) | a/b - c/d = 1/6\}

TASK 21

Given two rational numbers, represented as in Task 2 (e.g., \(\frac{3}{4} = \{\frac{3}{4}, \frac{6}{8}, \frac{9}{12}, \ldots\}\)) and two corresponding rationals, represented as in Task 11 (e.g., \(\frac{3}{4} = \{(a/b, c/d) | c/d - a/b = 3/4\}\)), show that this natural correspondence preserves the operation of addition.

RULE 21

Use Rule 3 to find the sum of the two given rationals. Next find the sum using Rule 13. Observe that the two sums correspond.

EXAMPLES 21

A. Given: \(\frac{3}{7}\) and \(\frac{3}{2}\).
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Answer:

By Rule 3, \{+3/-7, -3/-7, +6/+14, -6/-14, \ldots\} + \{+3/-2, -3/+2, +6/-4, -6/+4, \ldots\} = \{+15/-14, -15/+14, +30/-28, -30/+28, \ldots\} = -15/14

By Rule 13, \{(a/b, c/d) \mid a/b - c/d = 3/7\} + \{(a/b, c/d) \mid c/d - a/b = 3/2\} = \{(a/b, c/d) \mid c/d - a/b = 15/14\}; therefore the operation of addition is preserved for +3/7 and -3/2.


Answer:

By Rule 3, \{+2/-3, -2/+3, +4/-6, -4/+6, \ldots\} + \{+3/-4, -3/+4, +6/-8, -6/+8, \ldots\} = \{+17/-12, -17/+12, +34/-24, -34/+24, \ldots\} = -17/12.

By Rule 13, \{(a/b, c/d) \mid c/d - a/b = 2/3\} + \{(a/b, c/d) \mid c/d - a/b = 3/4\} = \{(a/b, c/d) \mid c/d - a/b = 17/12\}; therefore the operation of addition is preserved for -2/3 and -3/4.

EXERCISES 21

A. +2/5, +1/10
B. +3/2, -1/3
C. -5/6, +1/3
D. -4/5, -5/3

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TASK 22

Given two rational numbers, \(R_1\) and \(R_2\), show that the correspondence given in Task 21 preserves the operation of multiplication.

RULE 22

Find the product first by using Rule 4 and then by using Rule 14. Observe that the two products correspond.

EXAMPLES 22

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Answer:

By Rule 4, \({\frac{+3}{-3}, \frac{-4}{+3}, \frac{+8}{-6}, \frac{-8}{+6}, \ldots}\) \times \({\frac{-3}{-8}, \frac{-3}{+8}, \frac{+6}{-16}, \frac{-6}{+16}, \ldots}\) = \({\frac{+3}{-8}, \frac{-3}{+8}, \frac{+6}{-16}, \frac{-6}{+16}, \ldots}\) = \({\frac{+1}{2}, \frac{-1}{2}, \frac{+2}{4}, \frac{-2}{4}, \ldots}\) = \({\frac{+1}{2}, \frac{-1}{2}, \frac{+2}{4}, \frac{-2}{4}, \ldots}\)

By Rule 14, \(\{(a/b, c/d) \mid c/d - a/b = \frac{4}{3}\}\) \times \(\{(a/b, c/d) \mid c/d - a/b = \frac{3}{8}\}\) = \(\{(a/b, c/d) \mid a/b - c/d = \frac{1}{2}\}\)

Therefore the operation of multiplication is preserved for \(-\frac{4}{3}\) and \(-\frac{3}{8}\).

B. Given: \(+\frac{5}{6}\) and \(-\frac{3}{2}\).

Answer:

By Rule 4, \({\frac{+5}{+6}, \frac{-5}{-6}, \frac{+10}{+12}, \frac{-10}{-12}, \ldots}\) \times \({\frac{+3}{/2}, \frac{-3}{/2}, \frac{+6}{-4}, \frac{-6}{-4}, \ldots}\) = \({\frac{+5}{-4}, \frac{-5}{+4}, \frac{+10}{-8}, \frac{-10}{+8}, \ldots}\) = \(-\frac{5}{4}\).

By Rule 14, \(\{(a/b, c/d) \mid a/b - c/d = \frac{5}{6}\}\) \times \(\{(a/b, c/d) \mid c/d - a/b = \frac{3}{2}\}\) = \(\{(a/b, c/d) \mid c/d - a/b = \frac{5}{4}\}\)

Therefore the operation of multiplication is preserved for \(+\frac{5}{6}\) and \(-\frac{3}{2}\).

EXERCISES 22

A. \(+\frac{7}{3}, +\frac{9}{14}\)
B. \(+\frac{2}{5}, -\frac{15}{17}\)
C. \(-\frac{7}{2}, +\frac{4}{9}\)
D. \(-\frac{2}{3}, -\frac{4}{7}\)

Note: Tasks 21 and 22, together with the observation that the correspondence described in Task 21 is 1-1, show that the two ways of defining signed rationals (Tasks 2 and 11) lead to systems which are isomorphic.
EXERCISES 23

A. \( \frac{3}{4}, -\frac{2}{3} \)
B. \( -\frac{4}{7}, -\frac{3}{2} \)
C. \( -\frac{5}{2}, +\frac{1}{6} \)
D. \( +\frac{2}{7}, +\frac{3}{4} \)

\( \sqrt{ \text{TASK 24} } \)

Given two rational numbers, \( R_1 \) and \( R_2 \), determine whether or not the closure law for multiplication of rational numbers holds for \( R_1 \) and \( R_2 \).

RULE 24

Apply Rule 19, Chapter 5 to Rule 11, Chapter 5 and apply the derived rule to \( R_1 \) and \( R_2 \).

EXERCISES 24

A. \( -\frac{3}{2}, -\frac{4}{3} \)
B. \( +\frac{2}{5}, -\frac{7}{4} \)
C. \( +\frac{1}{6}, +\frac{2}{5} \)
D. \( -\frac{5}{7}, +\frac{4}{5} \)

\( \sqrt{ \text{TASK 25} } \)

Given three rational numbers, \( R_1, R_2 \), and \( R_3 \), determine whether or not the associative property for addition of rational numbers holds for \( R_1, R_2 \), and \( R_3 \).

RULE 25

Apply Rule 19, Chapter 5 to Rule 12, Chapter 5 and apply the derived rule to \( R_1, R_2 \), and \( R_3 \).

EXERCISES 25

A. \( +\frac{4}{3}, +\frac{3}{2}, +\frac{1}{6} \)
B. \( -\frac{2}{5}, -\frac{3}{10}, +\frac{2}{3} \)
C. \( +\frac{5}{6}, -\frac{3}{2}, +\frac{2}{3} \)
TASK 26

Given three rational numbers $R_1$, $R_2$, and $R_3$, determine whether or not the associative property for multiplication of rational numbers holds for $R_1$, $R_2$, and $R_3$.

RULE 26

Apply Rule 19, Chapter 5 to Rule 12, Chapter 5 and apply the derived rule to $R_1$, $R_2$, and $R_3$.

EXERCISES 26

A. $\frac{4}{5}$, $\frac{1}{3}$, $\frac{2}{1}$
B. $-\frac{7}{2}$, $\frac{4}{7}$, $-\frac{3}{2}$
C. $\frac{6}{7}$, $-\frac{14}{15}$, $\frac{5}{7}$
D. $-\frac{2}{3}$, $-\frac{3}{5}$, $-\frac{5}{6}$

TASK 27

Given two rational numbers $R_1$ and $R_2$, determine whether or not the commutative property for addition of rational numbers holds for $R_1$ and $R_2$.

RULE 27

Apply Rule 19, Chapter 5 to Rule 13, Chapter 5 and apply the derived rule to $R_1$ and $R_2$.

EXERCISES 27

A. $\frac{2}{5}$, $\frac{3}{2}$
B. $-\frac{7}{4}$, $\frac{5}{3}$
Given two rational numbers, \( R_1 \) and \( R_2 \), determine whether or not the commutative property for multiplication of rational numbers holds for \( R_1 \) and \( R_2 \).

RULE 28

Apply Rule 19, Chapter 5 to Rule 13, Chapter 5 and apply the derived rule to \( R_1 \) and \( R_2 \).

EXERCISES 28

A. \( \frac{-4}{3}, \frac{-6}{7} \)
B. \( \frac{+5}{9}, \frac{-3}{5} \)
C. \( \frac{+4}{11}, \frac{+3}{2} \)
D. \( \frac{-5}{6}, \frac{-6}{7} \)

Given a rational number \( R \), determine whether or not \( \frac{+0}{1} \) is an additive identity for \( R \).

RULE 29

Apply Rule 19, Chapter 5 to Rule 14, Chapter 5 and apply the derived rule to \( R \).

EXERCISES 29

A. \( \frac{-2}{9} \)
B. \( \frac{+7}{11} \)
C. \( \frac{-3}{5} \)
D. \( \frac{+6}{13} \)

(Note: \( \frac{+0}{1} \) is an additive identity for all rational numbers.)
\textbf{Task 30}

Given a rational number \( R \), determine whether or not \( +1/1 \) is a multiplicative identity for \( R \).

\textbf{Rule 30}

Apply Rule 19, Chapter 5 to Rule 14, Chapter 5 and apply the derived rule to \( R \).

\textbf{Exercises 30}

A. \( +4/5 \)
B. \( -3/7 \)
C. \( +9/2 \)
D. \( -5/11 \)

(Note: \( +1/1 \) is a multiplicative identity for all rational numbers.)

\textbf{Task 31}

Given two rational numbers \( x/y \) and \( x'/y' \), and given an identity for addition of rational numbers (e.g., \( +0/1 \)), determine whether or not \( x/y \) and \( x'/y' \) are inverses under addition.

\textbf{Rule 31}

Apply Rule 19, Chapter 5 to Rule 15, Chapter 5 and apply the derived rule to \( x/y \) and \( x'/y' \).

\textbf{Exercises 31}

A. \( -4/5, +4/5 \)
B. \( +7/6, -6/7 \)
C. \( -3/8, -3/8 \)
D. \( +9/5, -9/5 \)

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**TASK 32**

Given two non-zero rational numbers \( \frac{x}{y} \) and \( \frac{x'}{y'} \), and given an identity for multiplication of rational numbers (e.g., \( \frac{1}{1} \)), determine whether or not \( \frac{x}{y} \) and \( \frac{x'}{y'} \) are inverses under multiplication.

**RULE 32**

Apply Rule 19, Chapter 5 to Rule 15, Chapter 5 and apply the derived rule to \( \frac{x}{y} \) and \( \frac{x'}{y'} \).

**EXERCISES 32**

A. \( \frac{8}{3}, \frac{3}{8} \)  
B. \( \frac{-5}{7}, \frac{-7}{5} \)  
C. \( \frac{6}{11}, \frac{-11}{6} \)  
D. \( \frac{-8}{3}, \frac{-3}{8} \)

**TASK 33**

Given three rational numbers \( R_1, R_2, \) and \( R_3 \), determine whether or not the distributive property of multiplication over addition holds for \( R_1, R_2, \) and \( R_3 \).

**RULE 33**

Apply Rule 19, Chapter 5 to Rule 36, Chapter 5 and apply the derived rule to \( R_1, R_2, \) and \( R_3 \).

**EXERCISES 33**

A. \( \frac{4}{3}, \frac{-2}{7}, \frac{5}{2} \)  
B. \( \frac{-1}{3}, \frac{-2}{5}, \frac{-2}{3} \)  
C. \( \frac{4}{5}, \frac{-1}{10}, \frac{1}{2} \)  
D. \( \frac{-1}{4}, \frac{-2}{3}, \frac{-3}{5} \)

**TASK 34**

Given two rational numbers \( R_1 \) and \( R_2 \), determine whether or not \( R_1 \) is greater than \( R_2 \) (written \( R_1 > R_2 \)).
RULE 34

Find $R_1 - R_2$ (Rule 5). If $R_1 - R_2$ is positive, then $R_1 > R_2$. Otherwise, it is not.

EXAMPLES 34

A. Given: $\frac{3}{4}$ and $\frac{7}{5}$.
Answer:
$\frac{3}{4} - \frac{7}{5} = \frac{43}{20}$; therefore $\frac{3}{4} > \frac{7}{5}$.

B. Given: $\frac{-5}{16}$ and $\frac{-3}{8}$
Answer:
$\frac{-5}{16} - \frac{-3}{8} = \frac{1}{16}$; therefore $\frac{-5}{16} < \frac{-3}{8}$.

EXERCISES 34

A. $\frac{2}{3}, \frac{7}{12}$
B. $\frac{-4}{5}, \frac{1}{3}$
C. $\frac{3}{2}, \frac{-2}{7}$
D. $\frac{-3}{4}, \frac{-11}{16}$

TASK 35

Given a rational number $R$, show that there is a rational smaller than $R$.

RULE 35

Find $R - \frac{1}{1}$.

EXAMPLES 35

A. Given: $\frac{-2}{3}$.
Answer:
$\frac{-2}{3} - \frac{1}{1} = \frac{-5}{3}$

B. Given: $\frac{-15}{2}$
Answer:
$\frac{-15}{2} - \frac{1}{1} = \frac{-13}{2}$
EXERCISES 35

A. \( \frac{-12}{1} \)
B. \( \frac{+3}{7} \)
C. \( \frac{-21}{5} \)
D. \( \frac{+7}{5} \)

√TASK 36

Given two rationals \( R_1 \) and \( R_2 \), show that there is a rational between them.

RULE 36

Apply Rule 19, Chapter 5 to Rule 74, Chapter 7 and apply the derived rule to \( R_1 \) and \( R_2 \).

EXERCISES 36

A. \( \frac{-1}{4}, \frac{-3}{8} \)
B. \( \frac{+15}{2}, \frac{+29}{4} \)
C. \( \frac{-21}{5}, \frac{-4}{1} \)
D. \( \frac{+10}{3}, \frac{+11}{3} \)

√TASK 37

Given two integers \( x \) and \( y \), an operation \( * \), and the correspondence \( x \leftrightarrow \frac{x}{1} \) between the integers and the rational numbers, show that \( x * y \leftrightarrow \frac{x}{1} * \frac{y}{1} \).

RULE 37

Apply Rule 63, Chapter 8 to Rule 72, Chapter 7 and apply the derived rule to \( x \) and \( y \).

EXERCISES 37

A. \( \frac{-4}{2} \) and \( + \)
B. \( \frac{+3}{5} \) and \( x \)
C. \( \frac{-5}{7} \) and \( + \)
D. \( \frac{+6}{-3} \) and \( x \)
Given the positive rationals \( a/b = \{a, 2a, 3a, \ldots \} \) and \( c/d = \{c, 2c, 3c, \ldots \} \), and an operation \( \ast \), show that the correspondence \( a/b \leftrightarrow \ast a/b \) preserves the operation for \( a/b \) and \( c/d \).

**RULE 38**

Apply Rule 63, Chapter 8 to Rule 72, Chapter 7 and apply the derived rule to \( a/b \) and \( c/d \).

**EXERCISES 38**

A. \( \{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \ldots \}, \{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \ldots \} \) and \( x \).

B. \( \{\frac{3}{1}, \frac{6}{2}, \frac{9}{3}, \ldots \}, \{\frac{6}{5}, \frac{12}{10}, \frac{18}{15}, \ldots \} \) and \( + \).

C. \( \{\frac{4}{7}, \frac{8}{14}, \frac{12}{21}, \ldots \}, \{\frac{14}{9}, \frac{28}{18}, \frac{42}{27}, \ldots \} \) and \( x \).

D. \( \{\frac{6}{11}, \frac{12}{22}, \frac{18}{33}, \ldots \}, \{\frac{5}{3}, \frac{10}{6}, \frac{15}{9}, \ldots \} \) and \( + \).
SECTION 2. The System of Real Numbers

TASK 39

Given a number \( n \), find its square \( (n^2) \).

RULE 39

Find \( n \times n \).

EXAMPLES 39

A. Given: \( -3 \).

Answer:

\[ (-3)^2 = -3 \times -3 = 9. \]

B. Given: \( +\frac{7}{4} \).

Answer:

\[ \left( +\frac{7}{4} \right)^2 = +\frac{7}{4} \times +\frac{7}{4} = \frac{49}{16}. \]

EXERCISES 39

A. \( +\frac{4}{5} \)
B. \( -\frac{3}{7} \)
C. \( -9 \)
D. \( +\frac{2}{9} \)

---

TASK 40

Given a non-negative number \( n \), find the positive integer \( \sqrt{n} \) where \( \sqrt{n} \) is the positive square root of \( n \).

RULE 40

Select a non negative integer \( y_1 \leq \frac{n}{2} \). If \( y_1 \times y_1 = n \) then \( y_1 = \sqrt{n} \); otherwise \( y_1 \neq \sqrt{n} \). If \( y_1 \neq \sqrt{n} \), try another non negative integer \( y_2 \leq \frac{n}{2} \). Test to see if \( y_2 \times y_2 = n \) as before. Continue in this manner until a non negative integer \( x \) is found such that \( x \times x = n \) or you have convinced yourself that none exists (cf., Example B).
EXAMPLES 40

A. Given: 36

Answer:

Since \( 6 \times 6 = 36 \), \( \sqrt{36} = 6 \).

B. Given: 50

Answer:

Because \( 7 \times 7 = 49 \) and \( 8 \times 8 = 64 \), there is no non negative integer \( x \) such that \( x \times x = 50 \).

EXERCISES 40

A. 169
B. 225
C. 400
D. 121

\( \sqrt{ } \) \( \) TASK 41

Given a rule for generating \( a_n \), the \( n \)th or general term of a sequence, and a specific (stated) number \( k \), generate the \( k \)th term, \( a_k \), of the sequence.

RULE 41

Apply Rule 4, Chapter 6 to \( a_n \) (substitute \( k \) for \( n \) in \( a_n \)).

EXERCISES 41

A. \( a_n = \frac{7n}{2n - 1} \); find the 4\( \text{th} \) term.
B. \( a_n = \frac{3n}{3} \); find the 5\( \text{th} \) term.
C. \( a_n = 2n + n^2 \); find the 3\( \text{rd} \) term.
D. \( a_n = \frac{4n - 1}{5n} \); find the 8\( \text{th} \) term.
TASK 42

Given a rule for expressing $a_n$, generate the sequence.

RULE 42

Apply Rule 41 to generate $a_1$, $a_2$, $a_3$, ... (as many terms as desired). Write $(a_1, a_2, a_3, ..., a_n, ...)$. 

EXAMPLES 42

A. Given: $a_n = 3n^2$.

Answer:

$$a_1 = 3, a_2 = 12, a_3 = 27, (3, 12, 27, ..., 3n^2, ...)$$

B. Given: $a_n = \frac{4n + 2}{5n - 1}$

Answer:

$$a_1 = \frac{3}{2}, a_2 = \frac{10}{9}, a_3 = 1, a_4 = \frac{18}{19}, \left(\frac{3}{2}, \frac{10}{9}, 1, \frac{18}{19}, ..., \frac{4n + 2}{5n - 1}, ...\right)$$

EXERCISES 42

A. $a = \frac{1}{2n}$

B. $a = \frac{3n}{5n + 2}$

C. $a = 4n^3$

D. $a = 3 + 5n$


TASK 43

Given a rule for expressing $a_n$, and a number $k$, represent the first $k$ terms of the sequence as points on the number line.

RULE 43

Apply Rule 41 to generate $a_1$, $a_2$, $a_3$, ..., $a_k$. Next apply Rule 14, Chapter 8 to represent the non-negative rationals as points on the number line. Represent the
negative rational \(-\frac{a}{b}\) (e.g., \(-\frac{2}{3}\)) by a point the same distance to the left of 0 as \(\frac{a}{b}\) (e.g., \(\frac{+2}{3}\)) is to the right of 0.

**EXAMPLES 43**

A. Given: \(a_n = \frac{n}{n+1}, \quad k = 4\).

Answer:
\[
\begin{align*}
a_1 &= \frac{1}{2}, \quad a_2 = \frac{2}{3}, \quad a_3 = \frac{3}{4}, \quad a_4 = \frac{4}{5}.
\end{align*}
\]

B. Given: \(\frac{1 - 2n}{n}, \quad k = 3\).

Answer:
\[
\begin{align*}
a_1 &= -1, \quad a_2 = \frac{-3}{2}, \quad a_3 = \frac{-5}{3}.
\end{align*}
\]

**EXERCISES 43**

A. Represent the first 4 terms of \(a_n = 2n + 1\).
B. Represent the first 3 terms of \(a_n = \frac{3n - 2}{n + 1}\).
C. Represent the first 4 terms of \(a_n = 2 - \frac{n}{2}\).
D. Represent the first 5 terms of \(a_n = \frac{3n + 1}{n}\).
**TASK 44**

Given a rule for expressing $a_n$ and a number $L$, determine whether or not successive terms of the sequence approach (converge to) the given number $L$.

**RULE 44**

Consider the terms of the sequence as $n$ gets larger and larger -- i.e., as $n$ tends toward infinity (Rule 41). If these terms get progressively closer and closer to $L$, then the sequence approaches (converges to) $L$; otherwise it does not.

**EXAMPLES 44**

A. Given: The sequence whose $n^{th}$ term is $a_n = \frac{3n}{3n + 1}$. Determine whether or not this sequence approaches 1.

Answer:

As $n$ gets larger and larger, the terms get closer and closer to 1. (e.g., when $n = 100$, $a_n = 300/301$, when $n = 1000$, $a_n = 3000/3001$). Thus, the sequence approaches 1.

B. Given: The sequence whose $n^{th}$ term is $a_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even} \end{cases}$. Determine whether or not this sequence approaches 2.

Answer:

As $n$ gets larger and larger, the terms keep alternating 1, 2, 1, 2, etc. These terms do not get closer and closer to 2 and, thus, the sequence does not approach 2. [A similar argument shows that it does not approach 1 either.]

**EXERCISES 44**

A. $a_n = \frac{1}{3n}, \quad 0$

B. $a_n = \frac{n}{4n + 1}, \quad 4$

C. $a_n = \frac{n}{2}, \quad 153$

D. $a_n = \frac{3n + 1}{4n - 2}, \quad 3$

E. $a_n = ( - 1)^n, \quad -1$
TASK 45

Given two convergent sequences \((a_1, a_2, a_3, \ldots)\) and \((b_1, b_2, b_3, \ldots)\), use the definition of equivalence to determine whether or not the sequences are equivalent.

RULE 45

Form the sequence \((a_1 - b_1, a_2 - b_2, a_3 - b_3, \ldots)\). If this sequence approaches 0 (Rule 44), then the two given sequences are equivalent; otherwise they are not.

EXAMPLES 45

A. Given: \((2.1, 2.01, 2.001, 2.0001, \ldots)\) and \((3.9, 3.99, 3.999, 3.9999, \ldots)\).

Answer:

The difference sequence is \((-1.8, -1.98, -1.998, -1.9998, \ldots)\) which approaches -2. Therefore the two sequences are not equivalent.

B. Given: \((2.9, 2.99, 2.999, \ldots)\) and \((3.1, 3.01, 3.001, \ldots)\).

Answer:

The difference sequence is \((.2, .02, .002, \ldots)\) which approaches 0. Therefore the two sequences are equivalent.

EXERCISES 45

A. \((\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots)\), \((\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots)\)

B. \((-1.1, -1.01, -1.001, \ldots)\), \((-1, -1, -1, -1, \ldots)\)

C. \((.9, .99, .999, \ldots), (.1, .01, .001, .0001, \ldots)\)

D. \((5.2, 5.02, 5.002, \ldots)\), \((4, 4.1, 4.11, 4.111, \ldots)\)

---

TASK 46

Given a sequence \((a_1, a_2, a_3, \ldots, a_n, \ldots)\), determine whether or not it is distinguished.

RULE 46

Express each term as a decimal. If each term is formed from the preceding one by the addition of a single digit on the right, then the sequence is distinguished; otherwise it is not.
EXAMPLES 46

A. Given: (.4, .42, .428, .4285, .42857, .428571, .4285714, ...)

Answer:

This sequence is distinguished since each term is formed from the preceding one by adding one digit to the right.

B. Given: (1/2, 1/4, 1/8, 1/16, ...) = (.5, .25, .125, .0625, ...)

Answer:

This sequence is not distinguished since .0625 is not formed from the preceding term .125 by adding a single digit to the right of the "5" in .125.

EXERCISES 46

A. (3., 3.1, 3.12, 3.123, 3.1234, 3.12345, ...)
B. (1/3, 1/9, 1/27, 1/81, ...)
C. (.7, .77, .777, .7777, ...)
D. (2/5, 4/25, 8/125, 16/625, ...)

TASK 47

Given a real number R represented by a distinguished decimal sequence, represent the number as an infinite decimal.

RULE 47

Write the last term of the distinguished decimal sequence and write "..." following the last digit.

EXAMPLES 47

A. Given: (.2, .21, .212, .2121, .21212, ...)

Answer:

.21212 ...

B. Given: (1.4, 1.41, 1.414, 1.4142, ...).

Answer:

1.4142 ...
EXERCISES 47

A. (.9, .99, .999, .9999 ... )  
B. (-.2, -.25, -.250, -.2500, ... )  
C. (.4, .41, .412, .4124, .41241, .412412, ...)  
D. (7.1, 7.11, 7.111, 7.1111, ...) 

Note: As pointed out in the text, every distinguished decimal sequence converges to some real number R*. 

TASK 48 (Optional)

Given a distinguished decimal sequence and a rational number R, 0 < R < 1, find the term t_n such that the sequence converges to a real number R* less than t_n + R.

RULE 48

For each n = 1, 2, 3, ... take the difference t_{n+1} - t_n until t_{n+1} - t_n is less than R. Because the difference between that t_n and any term beyond it is less than R, the given sequence converges to some real number R* that is less than t_n + R.

EXAMPLES 48

A. Given: (3.2, 3.21, 3.214, 3.2143, ... ). R = .01. 

Answer: 

t_2 - t_1 = 3.21 - 3.2 = .01 
t_3 - t_2 = 3.214 - 3.21 = .004 < .01 

The difference between t_2 and any term beyond it is less than .01. Therefore, the given sequence converges to some real number R* that is less than 3.21 + .01 = 3.22.

B. Given: (.1, .12, .123, .1234, ..., .123456789, .1234567891, ... ), R = .001 

Answer: 

t_2 - t_1 = .12 - .1 = .02 
t_3 - t_2 = .123 - .12 = .003 
t_4 - t_3 = .1234 - .123 = .0004 < .001 

The difference between t_3 and any term beyond it is less than .001. Therefore, the given sequence converges to some real number R* that is less than .123 + .001 = .124.
EXERCISES 48

A. \( (2.4, 2.41, 2.412, 2.4123, 2.41235, \ldots), r = .1 \)
B. \( (.8, .83, .831, .8318, .83183, .831831, \ldots), r = .001 \)
C. \( (.6, .65, .654, .6543, \ldots, .654321, .6543216, \ldots), r = .0001 \)
D. \( (3.1, 3.12, 3.123, 3.1234, \ldots, 3.123456789, 3.1234567891, \ldots), r = .00001 \)

TASK 49

Given a real number \( R \), construct a sequence which represents it.

RULE 49

If \( R \) is rational, write \( (R + .1, R + .01, R + .001, R + .0001, \ldots) \). If \( R \) is not rational, choose a rational \( r_1 \) with \( R - 1 < r_1 < R \). Next choose a rational \( r_2 \) with \( r_1 < r_2 < R \) and a rational \( r_3 \) with \( r_2 < r_3 < R \). Continue this process and then write \( (r_1, r_2, r_3, r_4, \ldots) \).

Note: We could just as well have chosen the sequence \( (R - .1, R - .01, \ldots) \) when \( R \) is rational and \( (r_1, r_2, r_3, \ldots) \) where \( r_1 \) is chosen such that \( R < r_1 < r_1 - 1 \) etc., when \( R \) is not rational.

EXAMPLES 49

A. Given: .50000...

Answer:

\( (.5 + .1, .5 + .01, .5 + .001, .5 + .0001, \ldots) = (.6, .51, .501, .5001, \ldots) \).

B. Given: \( \sqrt{2} \).

Answer:

\[
\begin{array}{c|c}
 x & x \\
1.2 & \sqrt{2} \\
\end{array}
\]

Choose \( r_1 = 1.2, r_2 = 1.3, r_3 = 1.4, r_4 = 1.41, r_5 = 1.413, \ldots \)

(1.2, 1.3, 1.4, 1.41, 1.413, \ldots).

Note: A simple way of checking to see if the rational you pick is less than \( \sqrt{2} \) is to square it. If the square is less than 2, then the rational picked is less than \( \sqrt{2} \).
EXERCISES 49

A. \( \frac{1}{4} \)
B. \( \sqrt{3} \)
C. 7
D. \(-8.3\)

 TASK 50

Given a sequence \( (a_1, a_2, a_3, \ldots) \), which converges to a rational number (and hence represents the corresponding real number), represent the real number as an infinite decimal.

ANSWER 50

Use the terms of the given sequence to determine the rational number to which it converges. If the decimal representation of the rational number terminates, add 0's to it and adjoin "..." to the right of the 0's. If it does not terminate, express the repeating digits as in Rule 24 (ii), Chapter 7.

EXAMPLES 50

A. Given: \( (7.31, 7.301, 7.3001, 7.30001, \ldots) \).

Answer:

7.3000 ... since the terms of the sequence approach 7.3.

B. Given: \( (0.9, 0.99, 0.999, 0.9999, \ldots) \).

Answer:

1.000 ... since the terms of the sequence approach 1.

C. Given: \( (2/3 - 1/3, 2/3 - 1/9, 2/3 - 1/27, 2/3 - 1/81, \ldots; 2/3 - 1/3^n, \ldots) \).

Answer:

.6666 ... since the terms of the sequence approach \( 2/3 \).

EXERCISES 50

A. \( (1.3, 1/9, 1/27, 1/81, \ldots) \)
B. \( (2.9, 2.99, 2.999, 2.9999, \ldots) \)
C. \( (5 - 1/2, 5 - 1/4, 5 - 1/8, 5 - 1/16, \ldots) \)
D. (".5, ".25, ".125, ".0625, ... )

TASK 51

Given two real numbers $R_1^#$ and $R_2^#$, find the sum $R_1^# + R_2^#$.

RULE 51

Construct sequences $(a_1, a_2, a_3, ...)$ and $(b_1, b_2, b_3, ...)$ which represent $R_1^#$ and $R_2^#$, respectively (Rule 49). Add corresponding terms of the sequences to form the sum $(a_1 + b_1, a_2 + b_2, a_3 + b_3, ...)$. If the sum converges to a rational number, apply Rule 50.

EXAMPLES 51

A. Given: $4.000 ...$ and $\sqrt{2} = 1.41421...$

Answer:

Choose representatives $(4, 4.1, 4.01, 4.001, 4.0001, 4.00001, ...)$ and $(1, 1.4, 1.41, 1.414, 1.4142, 1.41421, ...) for $4.000 ...$ and $\sqrt{2}$ respectively. The sequence $(5, 5.5, 5.42, 5.415, 5.4143, 5.41422, ...)$ represents the sum.

B. Given: $3.000 ...$ and $2.000 ...$

Answer:

Choose representatives $(3.1, 3.01, 3.001, 3.0001, ...)$ and $(2.1, 2.01, 2.001, 2.0001, ...)$ for $3.000 ...$ and $2.000 ...$ respectively. The sequence $(5.2, 5.02, 5.002, 5.0002, ...)$ represents the sum (which by Rule 50 is $5.000 ...$).

C. Given: $1.5000 ...$ and ".02500 ...$

Answer:

Choose representatives $(1.6, 1.51, 1.501, 1.5001, ...)$ and "(.15, ".24, ".249, ".2499, ...)$ for $1.500 ...$ and ".02500 ...$ respectively. The sequence $(1.45, 1.27, 1.252, 1.2502, ...)$ represents the sum (which by Rule 50 is $1.2500 ...$).
EXERCISES 51

A. ~2.000 ..., 4.000 ...
B. 1.7000 ..., 3.8000 ...
C. ~.5000 ..., ~.7500 ...
D. 6.000 ..., ~1.000 ...

\[ \text{RULE 52} \]

Apply Rule 23, Chapter 5 to \( R^1 \) and \( R^2 \).

EXERCISES 52

A. ~2.000 ..., 7.000 ...
B. 1.333 ..., .2000 ...
C. 1.000 ..., .3000 ...
D. ~7.000 ..., ~3.000 ...

\[ \text{RULE 53} \]

Given two real numbers \( R^1 \) and \( R^2 \), find the product \( R^1 \times R^2 \).

\[ \text{EXAMPLES 53} \]

A. Given: \( 1/2 \) and 2.

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Answer:

1/2 is \(\left\{\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \ldots, \frac{n}{(2n+1)}\right\}\)
2 is \(\left\{\frac{3}{1}, \frac{5}{2}, \frac{7}{3}, \frac{9}{4}, \ldots, \frac{2n+1}{n}\right\}\)

The product is \((1, 1, 1, \ldots, n, 2n+D \times (2n+1)/n, \ldots)\) and by Rule 50 equals 1.000...

B. Given: \(\frac{1}{4}\) and \(+\frac{1}{2}\).

Answer:

\(\frac{1}{4}\) is \((\frac{1}{8}, \frac{3}{16}, \frac{5}{24}, \ldots, (1-2n)/8n, \ldots)\)
\(\frac{1}{2}\) is \((\frac{3}{7}, \frac{6}{13}, \frac{9}{19}, \ldots, 3n/(1+6n), \ldots)\)

The product is \((\frac{3}{56}, \frac{18}{208}, \frac{45}{456}, (1-2n)/(8n) \times (3n)/(1+6n), \ldots)\)
and by Rule 50, equals .125000....

EXERCISES 53

A. \(\frac{2}{3}, \frac{3}{2}\)
B. \(\frac{1}{4}, 6\)
C. \(3, -2\)
D. \(-1/3, -1/2\)

\(\sqrt{\text{TASK 54}}\)

Given two real numbers, \(R_1\) and \(R_2\), determine whether or not the closure property for addition of real numbers holds for \(R_1\) and \(R_2\).

RULE 54

Apply \(\Box\) Rule 19, Chapter 5 to Rule 11, Chapter 5 and apply the derived rule to \(R_1\) and \(R_2\).

EXERCISES 54

A. \(\ldots, 3.000 \ldots\)
B. \(\ldots, 2.000 \ldots\)
C. \(\ldots, \ldots, 5.000 \ldots\)
D. \(\ldots, 1.333 \ldots\)

\(\sqrt{\text{TASK 55}}\)

Given two real numbers, \(R_1\) and \(R_2\), determine whether or not the closure property for multiplication of real numbers holds for \(R_1\) and \(R_2\).
RULE 55

Apply Rule 19, Chapter 5 to Rule 11, Chapter 5 and apply the derived rule to $R_1^#$ and $R_2^#$. 

EXERCISES 55

A. $0.666 \ldots , 0.5000 \ldots$
B. $-3.000 \ldots , 2.333 \ldots$
C. $4.000 \ldots , -2.000 \ldots$
D. $-1.000 \ldots , -2.5000 \ldots$

\\

\(\sqrt{\text{TASK 56}}\)

Given three real numbers, $R_1^#$, $R_2^#$, and $R_3^#$, determine whether or not the associative property for addition of real numbers holds for $R_1^#$, $R_2^#$, and $R_3^#$. 

RULE 56

Apply Rule 19, Chapter 5 to Rule 12, Chapter 5 and apply the derived rule to $R_1^#$, $R_2^#$, and $R_3^#$. 

EXERCISES 56

A. $-2.000 \ldots , 4.000 \ldots , -3.000 \ldots$
B. $-0.75000 \ldots , -0.4000 \ldots , 1.000 \ldots$
C. $-0.25000 \ldots , 5.000 \ldots , -0.333 \ldots$
D. $-0.666 \ldots , 1.5000 \ldots , 0.75000 \ldots$

\\

\(\sqrt{\text{TASK 57}}\)

Given three real numbers $R_1^#$, $R_2^#$, and $R_3^#$, determine whether or not the associative property for multiplication of real numbers holds for $R_1^#$, $R_2^#$, and $R_3^#$. 

RULE 57

Apply Rule 19, Chapter 5 to Rule 12, Chapter 5 and apply the derived rule to $R_1^#$, $R_2^#$, and $R_3^#$. 

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EXERCISES 57

A. \( -0.666 \ldots, 3.000 \ldots, .5000\ldots \)
B. \( 1.000 \ldots, .25000 \ldots, -5.000 \ldots \)
C. \( -3.000 \ldots, -2.000 \ldots, -4.000 \ldots \)
D. \( .5000 \ldots, 1.333 \ldots, -.6000 \ldots \)

\( \sqrt{ } \) TASK 58

Given two real numbers, \( R_1^\# \) and \( R_2^\# \), determine whether or not the commutative property for addition of real numbers holds for \( R_1^\# \) and \( R_2^\# \).

RULE 58

Apply Rule 19, Chapter 5 to Rule 13, Chapter 5 and apply the derived rule to \( R_1^\# \) and \( R_2^\# \).

EXERCISES 58

A. \( -3.000 \ldots, 4.000 \ldots \)
B. \( .5000 \ldots, 3.000 \ldots \)
C. \( .75000 \ldots, -2.000 \ldots \)
D. \( 5.000 \ldots, -4.000 \ldots \)

\( \sqrt{ } \) TASK 59

Given two real numbers, \( R_1^\# \) and \( R_2^\# \), determine whether or not the commutative property for multiplication of real numbers holds for \( R_1^\# \) and \( R_2^\# \).

RULE 59

Apply Rule 19, Chapter 5 to Rule 13, Chapter 5 and apply the derived rule to \( R_1^\# \) and \( R_2^\# \).

EXERCISES 59

A. \( 3.000 \ldots, -4.000 \ldots \)
B. \( .5000 \ldots, 1.333 \ldots \)
C. \( -2.000 \ldots, -1.000 \ldots \)
D. \( .75000 \ldots, -4.000 \ldots \)
TASK 60

Given a real number, \( R^\# \), determine whether or not the real number 0.000 \( \ldots \) is an additive identity for \( R^\# \).

RULE 60

Apply Rule 19, Chapter 5 to Rule 14, Chapter 5 and apply the derived rule to \( R^\# \).

EXERCISES 60

A. \( 1.5000 \ldots \)
B. 5
C. \( \frac{1}{4} \)
D. \( \frac{1}{2} \)

Note: 0.000 \( \ldots \) is the additive identity for all real numbers.

TASK 61

Given a real number \( R^\# \), determine whether or not the real number 1.000 \( \ldots \) is a multiplicative identity for \( R^\# \).

RULE 61

Apply Rule 19, Chapter 5 to Rule 14, Chapter 5 and apply the derived rule to \( R^\# \).

EXERCISES 61

A. \( 0.666 \ldots \)
B. \( 3.000 \ldots \)
C. \( 4.000 \ldots \)
D. \( 0.5000 \ldots \)

Note: 1.000 \( \ldots \) is the multiplicative identity for all real numbers.

TASK 62

Given two real numbers \( R^1 \) and \( R^2 \), and an identity for addition of real numbers
(e.g., 0.000 ... ), determine whether or not \( R_1^\# \) and \( R_2^\# \) are inverses under addition.

RULE 62

Apply Rule 19, Chapter 5 to Rule 15, Chapter 5 and apply the derived rule to \( R_1^\# \) and \( R_2^\# \).

EXERCISES 62

A. 2.000 ..., -2.000 ...
B. .25000 ..., .25000 ...
C. 3.000 ..., 2.000 ...
D. 5.000 ..., .2000 ...

\[ \text{________} \cdot \text{________} \]

\( \sqrt{\text{TASK 63}} \)

Given two non-zero real numbers \( R_1^\# \) and \( R_2^\# \), and an identity for multiplication of real numbers (e.g., 1.000 ... ), determine whether or not \( R_1^\# \) and \( R_2^\# \) are inverses under multiplication.

RULE 63

Apply Rule 19, Chapter 5 to Rule 15, Chapter 5 and apply the derived rule to \( R_1^\# \) and \( R_2^\# \).

EXERCISES 63

A. .6000 ..., 1.666 ...
B. -2.000 ..., -.5000 ...
C. 7.000 ..., -7.000 ...
D. 1.333 ..., .75000 ...

\[ \text{________} \cdot \text{________} \]

\( \sqrt{\text{TASK 64}} \)

Given three real numbers \( R_1^\# \), \( R_2^\# \), and \( R_3^\# \), determine whether or not the distributive property of multiplication over addition holds for \( R_1^\# \), \( R_2^\# \), and \( R_3^\# \).

RULE 64

Apply Rule 19, Chapter 5 to Rule 36, Chapter 5 and apply the derived rule to \( R_1^\# \), \( R_2^\# \), and \( R_3^\# \).
EXERCISES 64

A. 3.000 ..., .5000 ..., -1.5000...
B. -2.000 ..., 1.000 ..., 3.000 ...
C. .333 ..., .75000 ..., .25000 ...
D. -1.000 ..., 2.000 ..., -3.000 ...

TASK 65

Given the rationals R₁ and R₂, an operation *, and the correspondence R → R₆ between the rationals and the reals, show that R₁ * R₂ → R₆ * R₆.

RULE 65

Apply Rule 63, Chapter 8 to Rule 72, Chapter 7 and apply the derived rule to R₁ and R₂.

EXAMPLES 65

A. Given: 2, 1/4, and +.
Answer:

2 ↔ 2.000 ... and 1/4 ↔ .25000 ...
2 + 1/4 = 2 1/4 and 2.000 ... + .25000 ... = 2.25000 ..., Observe that 2 1/4 ↔ 2.25000 ...

B. Given: 2/3, 3/1, and x.
Answer:

2/3 ↔ (2/5, 4/8, 6/11, ..., 2n/(3n+2),...) and 3/1 ↔ (4, 7/2, 10/3, ..., (3n+1)/n,...).
3/1 x 2 = 2 and (3/5, 4/8, 6/11, ..., 2n/(3n+2),...) x (4, 7/2, 10/3, ..., (3n+1)/n,...)
= (8/5, 7/4, 20/11, ..., (6n²+2n)/(3n²+2n),...).
Observe that 2 ↔ (8/5, 7/4, 20/11, ..., (6n²+2n)/(3n²+2n),...).

EXERCISES 65

A. +3/2, -1/4, and +
B. -2/1, +4/1, and x
C. +1/3, -3/4, and +
D. -3/1, -2/1, and x
SECTION 3. Further Extensions

TASK 66

Given two vectors \((a_1, a_2, \ldots, a_n)\) and \((b_1, b_2, \ldots, b_n)\), find their sum.

RULE 66

Add the given vectors componentwise -- i.e., the sum is given by \((a_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots, a_n + b_n)\).

EXAMPLES 66

A. Given: \((3, -2)\) and \((4, 1)\).

Answer:

\((3, -2) + (4, 1) = (7, -1)\).

B. Given: \((1, 0, 2, 3)\) and \((-1, 3, -2, 4)\).

Answer:

\((1, 0, 2, 3) + (-1, 3, -2, 4) = (0, 3, 0, 7)\).

EXERCISES 66

A. \((2, 5, 3), (4, -2, 1)\)
B. \((1, 0, 2, 5, 6), (-2, -4, 0, 3, 5)\)
C. \((2, 5), (5, 2)\)
D. \((-2, 4, 1, 3), (3, 1, 0, -2)\)

---

TASK 67

Given a vector \((a_1, a_2, \ldots, a_n)\) and a scalar \(c\), find the scalar product of \(c\) with the vector.

RULE 67

Multiply each component of the vector by \(c\) -- i.e., \(c \cdot (a_1, a_2, \ldots, a_n) = (c \cdot a_1, c \cdot a_2, c \cdot a_3, \ldots, c \cdot a_n)\).
EXAMPLES 67

A. Given: (3, -2, 1) and 5.

Answer:

5 ⋅ (3, -2, 1) = (15, -10, 5).

B. Given: (1/2, -4/3, 2, 1) and -6.

Answer:

-6 ⋅ (1/2, -4/3, 2, 1) = (-3, 8, -12, -6).

EXERCISES 67

A. (4, -2/3), 2
B. (-3, 1, 4, 1/2), 1/3
C. (2, 5, -3), 3
D. (-4, 6/5, 3/4, 2, 5), 10

TASK 68

Given a complex number in the form \( a + bi \), represent it as an ordered pair and vice versa.

RULE 68

Write \( (a, b) \) as \( a + bi \) or \( a + bi \) as \( (a, b) \).

EXAMPLES 68

A. Given: (3, 2).

Answer:

(3, 2) = 3 + 2i.

B. Given: -1 + 7i.

Answer:

-1 + 7i = (-1, 7).
EXERCISES 68

A. \((4, -2)\)
B. \(-3 + 5i\)
C. \((-3, -7)\)
D. \(5 + 9i\)

---

TASK 69

Given two complex numbers \((a, b)\) and \((c, d)\), find the sum.

RULE 69

Add the given numbers componentwise -- i.e., \((a, b) + (c, d) = (a + c, b + d)\).

EXAMPLES 69

A. Given: \((-2, 1/3)\) and \((4, 3)\).

Answer:

\((-2, 1/3) + (4, 3) = (2, 10/3)\).

B. Given: \((1/5, 1/4)\) and \((-4/5, 3/4)\).

Answer:

\((1/5, 1/4) + (-4/5, 3/4) = (-3/5, 1)\).

EXERCISES 69

A. \((-2, 5), (3, 4)\)
B. \((1/4, -3), (3, -2/5)\)
C. \((-1, 1), (2, 3)\)
D. \((0, 0), (-3, 5)\)

---

TASK 70

Given two complex numbers \((a, b)\) and \((c, d)\), find their product.

RULE 70

Perform the operations indicated by the following: \((a, b) \times (c, d) = \)
((a x c) - (b x d), (a x d) + (b x c)).

EXAMPLES 70

A. Given: (3, 2) and (-2, 5).

Answer:

(3, 2) x (-2, 5) = ((3 x -2) - (2 x 5), (3 x 5) + (2 x -2)) =
(-6 - 10, 15 + -4) = (-16, 11).

B. Given: (1, 3) and (-2, -4).

Answer:

(1, 3) x (-2, -4) = ((1 x -2) - (3 x -4), (1 x -4) + (3 x -2)) =
(-2 - -12, -4 + -6) = (10, -10).

EXERCISES 70

A. (1/2, 2), (4.3, 0)
B. (1, 0), (5, 7)
C. (-2, -4), (0, 3)
D. (9, 7), (0, 1)

√TASK 71

Given two complex numbers (a, b) and (c, d), determine whether or not the closure law for addition of complex numbers holds for (a, b) and (c, d).

RULE 71

Apply Rule 19, Chapter 5 to Rule 11, Chapter 5 and apply the derived rule to (a, b) and (c, d).

EXERCISES 71

A. (3, 1), ( -4, 2)
B. (1/2, 3), (7/2, -4/3)
C. (0, 1), (3, -2)
D. (3/5, 2), (7/5, 1/2)
**V TASK 72**

Given two complex numbers \((a, b)\) and \((c, d)\), determine whether or not the closure law for multiplication of complex numbers holds for \((a, b)\) and \((c, d)\).

**RULE 72**

Apply Rule 19, Chapter 5 to Rule 11, Chapter 5 and apply the derived rule to \((a, b)\) and \((c, d)\).

**EXERCISES 72**

A. \((2, 4), (3, 1)\)
B. \((1/2, 3), (2/3, 5)\)
C. \((-1, 0), (0, -2)\)
D. \((3, -4), (-2, 7)\)

---

**V TASK 73**

Given three complex numbers \((a, b)\), \((c, d)\), and \((e, f)\), determine whether or not the associative property for addition of complex numbers holds for \((a, b)\), \((c, d)\), and \((e, f)\).

**RULE 73**

Apply Rule 19, Chapter 5 to Rule 12, Chapter 5 and apply the derived rule to \((a, b)\), \((c, d)\), and \((e, f)\).

**EXERCISES 73**

A. \((-4, 2), (3, 1), (5, -7)\)
B. \((1/2, 3), (2/3, -4), (4, 0)\)
C. \((0, -2), (3, 1/4), (-1, -1)\)
D. \((5, 3), (5/3, 2), (1, 0)\)

---

**V TASK 74**

Given three complex numbers, \((a, b)\), \((c, d)\), and \((e, f)\), determine whether or not the associative property for multiplication of complex numbers holds for \((a, b)\), \((c, d)\), and \((e, f)\).
RULE 74

Apply Rule 19, Chapter 5 to Rule 12, Chapter 5 and apply the derived rule to
(a, b), (c, d), and (e, f).

EXERCISES 74

A. (5, 3), (2, 0), (3, -1)
B. (-2, 1/3), (3, 1), (1/2, 0)
C. (-4, -3), (-1/3, 2), (0, 1)
D. (1, 2), (3, 4), (2, 1)

√ TASK 75

Given two complex numbers (a, b) and (c, d), determine whether or not the commutative property for addition of complex numbers holds for (a, b) and (c, d).

RULE 75

Apply Rule 19, Chapter 5 to Rule 13, Chapter 5 and apply the derived rule to
(a, b) and (c, d).

EXERCISES 75

A. (2, 2), (-3, 4)
B. (-1, 5), (3, -2)
C. (1/2, 1), (1/3, -5)
D. (-4, 7), (1/3, 2)

√ TASK 76

Given two complex numbers (a, b) and (c, d), determine whether or not the commutative property for multiplication of complex numbers holds for (a, b) and (c, d).

RULE 76

Apply Rule 19, Chapter 5 to Rule 13, Chapter 5 and apply the derived rule to
(a, b) and (c, d).

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**EXERCISES 76**

A. \((-7, \frac{1}{3}), (3, -4)\)
B. \((0, 2), (1, 3)\)
C. \((-3, -4), (5, 2)\)
D. \((4, -2), (3, 5)\)

---

**VTASK 77**

Given a complex number \((a, b)\), determine whether or not the complex number \((0, 0)\) is an additive identity for \((a, b)\).

**RULE 77**

Apply Rule 19, Chapter 5 to Rule 14, Chapter 5 and apply the derived rule to \((a, b)\).

**EXERCISES 77**

A. \((4, -1)\)
B. \((-7, -5)\)
C. \((3, 2)\)
D. \((-2, 4)\)

Note: \((0, 0)\) is an additive identity for all complex numbers.

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**VTASK 78**

Given a complex number \((a, b)\), determine whether or not the complex number \((1, 0)\) is a multiplicative identity for \((a, b)\).

**RULE 78**

Apply Rule 19, Chapter 5 to Rule 14, Chapter 5 and apply the derived rule to \((a, b)\).

**EXERCISES 78**

A. \((14, 3)\)
B. \((-\frac{1}{3}, 2)\)
C. \((5, \frac{1}{4})\)
D. \((-2, -5)\)
Note: \((1, 0)\) is a multiplicative identity for all complex numbers.

\[
\]

\sqrt{\text{TASK 79}}

Given the complex numbers \((a, b)\) and \((c, d)\), and given an identity for addition of complex numbers (e.g., \((0, 0)\)), determine whether or not \((a, b)\) and \((c, d)\) are inverses under addition.

\text{RULE 79}

Apply \text{Rule 19, Chapter 5} to \text{Rule 15, Chapter 5} and apply the derived rule to \((a, b)\) and \((c, d)\).

\text{EXERCISES 79}

A. \((2, -3), (2, 3)\)
B. \((5, 7), (-5, -7)\)
C. \((-6, -2), (6, -2)\)
D. \((0, 4), (0, -4)\)

\sqrt{\text{TASK 80}}

Given two complex numbers \((a, b) \neq (0, 0)\), and \((c, d) \neq (0, 0)\), and given an identity for multiplication of complex numbers (e.g., \((1, 0)\)), determine whether or not \((a, b)\) and \((c, d)\) are inverses under multiplication.

\text{RULE 80}

Apply \text{Rule 19, Chapter 5} to \text{Rule 15, Chapter 5} and apply the derived rule to \((a, b)\) and \((c, d)\).

\text{EXERCISES 80}

A. \((4, -3), (4/25, 3/25)\)
B. \((2, 5), (2/29, -5/29)\)
C. \((-3, -7), (-1/3, 1/7)\)
D. \((-6, 0), (-1/6, 0)\)
\(\sqrt{\text{Task 81}}\)

Given three complex numbers \((a, b), (c, d),\) and \((e, f)\), determine whether or not the distributive property of multiplication over addition holds for \((a, b), (c, d),\) and \((e, f).\)

**Rule 81**

Apply \(\oplus\) Rule 19, Chapter 5 to Rule 36, Chapter 5 and apply the derived rule to \((a, b), (c, d),\) and \((e, f).\)

**Exercises 81**

A. \((3, 2), (1, -7), \frac{1}{2}, 0)\)
B. \((-\frac{1}{4}, 3), (0, 1), (-2, -2))\)
C. \((5, 1), (4, 6), (-3, -2))\)
D. \((0, 1), (2, 3), (1/3, 5))\)

\[\]  

**Task 82**

Given a complex number \((a, b)\) represent it graphically.

**Rule 82**

Move \(a\) units along the horizontal axis (right if sign of \(a\) is +, left if sign of \(a\) is -) and then \(b\) units up (if sign of \(b\) is +) or down (if sign of \(b\) is -) on a line parallel to the vertical axis (through \(a\). This point label \((a, b).\)

**Examples 82**

A. Given: \((4, -2))\)

Answer:

\[\]
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B. Given: (3, 2)

Answer:

EXERCISES 82

A. (2, -4)
B. (3, 1)
C. (2, -1)
D. (-4, 3)

\[ (3, 2) \]

VTASK 83

Given the real numbers \( a \) and \( b \), an operation \( * \), and the correspondence \( a \mapsto (a, 0) \) between the real numbers and the complex numbers, show that \( a * b \mapsto (a, 0) \) * \( (b, 0) \).

RULE 83

Apply Rule 63, Chapter 8 to Rule 72, Chapter 7 and apply the derived rule to \( a, b, \) and \( \ast \).

EXERCISES 83

A. 5, -2, and +
B. -3, 1/2 and x
C. 1/5, 6 and +
D. -4, -1, and x

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